CLASSIFICATION OF LEAVITT PATH ALGEBRAS USING ALGEBRAIC K-THEORY

A Dissertation

Presented to

the Faculty of the Department of Mathematics

University of Houston

In Partial Fulfillment

of the Requirements for the Degree

Doctor of Philosophy

By

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May 2015

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Acknowledgements

This thesis consists of joint work with not only my advisor Dr. Mark Tomforde but also James Gabe and Dr. Efren Ruiz. I am grateful for being able to take part in their work. James Gabe has a tremendous amount of mathematical skill, and his ability to notice details astounds me. Dr. Ruiz is also possessed of an incredible mathematical talent. I am humbled by the opportunity to work with them.

I owe a great deal to Dr. Tomforde for his time, patience, and endless help. He put many, many hours into teaching me advanced mathematics, as well as showing me around the mathematical community. Thanks to him I attended conferences, composed and gave talks, wrote and submitted a paper, met many other accomplished mathematicians, developed my job portfolio, and more. I cannot thank Dr. Tomforde enough, and would not have accomplished this without him.

My family provided much assistance, both tangible (a car) and intangible (motivation) to go to graduate school. My mom and dad, Carol and Gary, are the greatest parents I could have, giving me encouragement, advice, cards, gifts, support, prayers, and love. My brother Trevor also gave great encouragement and great online multiplayer Nintendo battles. My grandmother Doris ("Granny B" to us) gave constant encouragement and love, as well as helping financially in ways large and small.

I believe my wife Grace should have her name on my diploma as much as I should. Her constant love, service, and support throughout my graduate career was and is essential and I would not have made it without her.

Grace and I are both thankful for our church family at Providential Baptist

Church, which has been our church home since the first week we lived in the Houston area. During my entire graduate career, our church gave us wise council, love and encouragement, prayer and guidance, fun and laughs, and everything needed to make it a family away from home.

Finally and most importantly I thank God for it all. That such a person as I ever produced a dissertation in mathematics is evidence enough to me of His work in my life. Indeed, without God in the equation it is inconceivable to me that I would accomplish anything more than primordial ooze could accomplish, if I existed at all. For surrounding me with all the people I have thanked above, providing me the opportunities to attend and work at the University of Houston, and giving me whatever abilities I possess, in addition to meeting my daily and hourly needs, and on top of already making the ultimate sacrifice, I thank the Lord my God.

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Abstract

We show that the long exact sequence for the K-theory of Leavitt path algebras over row-finite graphs, discovered by Ara, Brustenga, and Cortiñas, extends to Leavitt path algebras of arbitrary countable graphs. Using this long exact sequence, we compute explicit formulas for the higher K-groups of Leavitt path algebras in several situations, including all of the K-groups of Leavitt path algebras over finite fields and algebraically closed fields. We then focus on the classification up to Morita equivalence of purely infinite simple unital Leavitt path algebras over countably infinite graphs. The K_0-group and the K_1-group are not sufficient to classify these Leavitt path algebras when the underying field is the rational numbers. We prove that when the underlying field is a number field (which includes the rational numbers), then these Leavitt path algebras are classified up to Morita equivalence by the K_0-group and the K_6-group.

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CHAPTER 1

Introduction

When we first encounter $B(H)$, the bounded operators on an infinite-dimensional Hilbert space H, we have a wealth of directions that can be investigated. One option is to consider subalgebras of $B(H)$, and in doing so we encounter examples that include continuous complex-valued functions on compact spaces, matrix algebras, the compact operators, and L^∞-spaces, to name a few. Each of these algebras may be identified with a $*$-subalgebra of $B(H)$ that is closed in norm; that is, each of these is a C^*-*algebra*. These examples illustrate that C^*-algebras interact with topology, linear algebra, and measure theory.

The theory of C^*-algebras extends well beyond these examples, however. For

instance, C^*-algebras are tools in the Haag-Kastler axiomatization of local quantum field theory used to describe systems in quantum mechanics. Two Fields medalists used C^*-algebras in their award-winning work: Alain Connes applied C^*-algebras to differential geometry, and Vaughan Jones related von Neumann algebras (a class of C^*-algebras) to solving problems in knot theory.

C^*-algebra theory also plays a role in symbolic dynamics. In our first course in Functional Analysis we learn that in infinite dimensions, there are one-to-one operators that are not onto (i.e., there are isometries that are not unitaries). The prototypical example of such an operator is the unilateral shift. Coburn's theorem tells us that any C^*-algebra generated by a single nonunitary isometry is isomorphic to the C^*-algebra generated by the unilateral shift. This is an amazing result that has applications to the study of Hardy spaces and Fredholm operators.

What about the C^*-algebra generated by n proper isometries? Provided that they have mutually orthogonal ranges, the C^*-algebra they generate is unique and is known as the Cuntz algebra $\mathcal{O}_n$. The Cuntz algebras are fundamental C^*-algebras that arise in the study of tensor products, K-theory, and classification of C^*-algebras. More generally, we may consider n mutually orthogonal *partial* isometries, which unlike isometries have nontrivial kernels. A partial isometry is, by definition, an isometry when restricted to the orthogonal complement of its kernel. We have to indicate how these partial isometries are related to each other, and we do this using an $n \times n$ matrix A. The C^*-algebra generated by n mutually orthogonal partial isometries that satisfy relations determined by A is called the Cuntz-Krieger algebra $\mathcal{O}_A$.

We may generalize even further by viewing the matrix A as an adjacency matrix of a directed graph. Within the directed graph, we associate each edge with a partial isometry and each vertex with a projection. We place no restriction on the number of vertices and edges, and the directed graph provides information on how the partial isometries and projections are related. The C^*-algebra generated by these mutually orthogonal partial isometries and projections is called the *graph C^*-algebra of E*.

Definition Let $E = (E^0, E^1, r, s)$ be a directed graph, where E^0 denotes the set of vertices, E^1 denotes the set of edges, and $r, s : E^1 \to E^0$ denote the range and source maps. The *graph C^*-algebra* $C^*(E)$ is the universal C^*-algebra generated by mutually orthogonal projections $\{P_v : v \in E^0\}$ and partial isometries $\{S_e : e \in E^1\}$ with mutually orthogonal ranges that satisfy the relations:

1. $S_e^* S_e = P_{r(e)}$ for all $e \in E^1$
2. $P_v = \sum_{e \in E^1 : s(e) = v} S_e S_e^*$ when $v \in E^0$ with $0 < |s^{-1}(v)| < \infty$
3. $S_e S_e^* \leq P_{s(e)}$ for all $e \in E^1$

Graph C^*-algebras are not the only objects "built" from directed graphs. In symbolic dynamics, a directed graph is used in the construction of shift spaces of finite type. We begin with an alphabet $\mathcal{A}$, which is a collection of symbols. Then $\mathcal{A}^{\mathbb{Z}}$ consists of bi-infinite sequences of symbols in $\mathcal{A}$. Usually we wish to restrict the kinds of sequences allowed, and we do this by specifying a set of forbidden blocks

(that is, a set of forbidden finite sequences). Then the bi-infinite sequences that do not contain any forbidden blocks form a subset of $\mathcal{A}^{\mathbb{Z}}$ called a shift space. If the set of forbidden blocks is finite, then we call the shift space a *shift of finite type*. Another way to construct a shift of finite type is to use a directed graph, where each edge represents a symbol in the alphabet, and the bi-infinite sequences in the shift space correspond to bi-infinite *paths* (or *walks*) in the graph. A shift space constructed in this way is called an *edge shift*.

Though edge shifts seem to be a special case of shifts of finite type, it turns out that every shift of finite type can be recoded to an edge shift [18, Theorem 2.3.2]. Certain concepts for graph C^*-algebras and shift spaces coincide. For example, the graph C^*-algebra $C^*(E)$ is simple if and only if the shift space associated to E is irreducible, and the Bowen-Franks group turns out to be the K_0-group of the graph C^*-algebra. Each theory has applications to the other.

Graph C^*-algebras not only vastly generalize the Cuntz-Krieger algebras with ties to symbolic dynamics but also capture other important classes of C^*-algebras as well, including finite-dimensional C^*-algebras, the algebra of compact operators on a Hilbert space, the algebra of continuous functions on the circle, many AF-algebras (all of them up to strong Morita equivalence), and many Kirchberg algebras.

Another wonderful aspect of graph C^*-algebras is the fact that the graph not only determines the relations the generators satisfy, but also reflects the structure of the graph C^*-algebra. Thus we may formulate algebraic properties, such as having a unit, being simple, and being purely infinite, in terms of properties of the graph, allowing us to look at E and "see" the structure of $C^*(E)$. Moreover, graph C^*-algebras are

amenable to classification and the invariants used to classify are relatively easy to compute.

The success of graph C^*-algebras inspired algebraists to create purely algebraic versions called *Leavitt path algebras* [2]. In 1962, a few years before Cuntz algebras were introduced, W. G. Leavitt constructed a class of k-algebras to show the existence of rings of arbitrary module type [16, Section 3]. These algebras are now known as *Leavitt algebras* and denoted $L_{\mathsf{k}}(1,n)$, where $n \in \mathbb{N}$ and k is a field. Both $L_{\mathsf{k}}(1,n)$ and $\mathcal{O}_n$ are simple (shown in [17] and [8] respectively). Furthermore, $\mathcal{O}_n$ may be viewed as the completion (in an appropriate norm) of $L_{\mathbb{C}}(1,n)$ over the field $\mathbb{C}$. For these reasons, we may view $\mathcal{O}_n$ as an "analytic analogue" of $L_{\mathsf{k}}(1,n)$. Leavitt path algebras, then, complete the analogy as "algebraic analogues" of graph C^*-algebras.

Definition Let $E = (E^0, E^1, r, s)$ be a graph, and let k be a field. We let $(E^1)^*$ denote the set of formal symbols $\{e^* : e \in E^1\}$. The *Leavitt path algebra of E with coefficients in* k, denoted $L_{\mathsf{k}}(E)$, is the free associative k-algebra generated by a set $\{v : v \in E^0\}$ of pairwise orthogonal idempotents, together with a set $\{e, e^* : e \in E^1\}$ of elements, modulo the ideal generated by the following relations:

1. $s(e)e = er(e) = e$ for all $e \in E^1$
2. $r(e)e^* = e^*s(e) = e^*$ for all $e \in E^1$
3. $e^*f = \delta_{e,f}\, r(e)$ for all $e, f \in E^1$
4. $v = \sum_{\{e \in E^1 : s(e) = v\}} ee^*$ whenever $v \in E^0_{\text{reg}}$.

While their definitions have similarities, these classes have important differences: The graph C^*-algebra $C^*(E)$ has a norm with respect to which it is complete, while $L_{\mathsf{k}}(E)$ has no norm and no notion of completeness. Furthermore, one considers C^*-algebras over the complex number field $\mathbb{C}$, whereas one considers Leavitt path algebras over general fields k. Finally, it is a nontrivial fact that if $\mathsf{k} = \mathbb{C}$, then $L_{\mathbb{C}}(E)$ is a dense $*$-subalgebra of $C^*(E)$ [26].

Surprisingly, Leavitt path algebras and graph C^*-algebras have numerous similarities beyond their definitions, and indeed several parallel results hold for both $L_{\mathsf{k}}(E)$ and $C^*(E)$. Perhaps more surprisingly, these similar results require very different methods to prove—algebraic techniques for $L_{\mathsf{k}}(E)$ and analytic techniques for $C^*(E)$—and neither collection of results implies the other.

The focus here is on Leavitt path algebras and their classification. The means of classification of Leavitt path algebras is motivated by the success of graph C^*-algebra classification. For general C^*-algebras, the most popular invariant for classification is analytic K-theory, which is similar in theory to homology used in algebraic topology. Given a C^*-algebra A, one constructs an abelian group $K_n(A)$ for each $n \in \mathbb{Z}$. Due to a phenomenon known as Bott periodicity, the even K-groups are isomorphic to each other, and the odd K-groups are isomorphic to each other. So, only $K_0(A)$ and $K_1(A)$ are needed. These groups are difficult to define and in general difficult to compute, but for a graph C^*-algebra $C^*(E)$ the computation of $K_0(C^*(E))$ and $K_1(C^*(E))$ is relatively easy.

The most popular invariant for the classification of Leavitt path algebras is algebraic K-theory, which is the algebraic analogue of analytic K-theory. In general, however, algebraic K-groups have no periodicity, so *a priori* one might need all the algebraic K-groups for classification. To make matters worse, algebraic K-theory is notoriously difficult to define, much less compute (in fact, not even all the algebraic K-groups of $\mathbb{Z}$ are known).

Fortunately, algebraic K-theory has provided a useful invariant in the classification of Leavitt path algebras. Abrams, Louly, Pardo, and Smith showed that if E is a finite, strongly connected graph, then the only K-group needed for classification is $K_0(L_{\mathsf{k}}(E))$ [3]. For the next case, Ruiz and Tomforde showed that if a strongly connected graph E has only finitely many vertices but infinitely many edges, then $L_{\mathsf{k}}(E)$ is classified by $K_0(L_{\mathsf{k}}(E))$ together with the number of singular vertices (i.e., the number of vertices that emit either no edges or infinitely many) in E [22]. This is a very nice invariant, but the number of singular vertices depends on the graph, and hence on the way the algebra is presented. If we wish to extend the result to more general algebras, we would like an invariant described entirely in terms of algebraic properties. So, we seek to replace the number of singular vertices with an invariant defined entirely in terms of algebraic properties of $L_{\mathsf{k}}(E)$, such as other algebraic K-groups. Ruiz and Tomforde discovered that $K_0(L_{\mathsf{k}}(E))$ and $K_1(L_{\mathsf{k}}(E))$ do provide a complete invariant for classification when the underlying field k has a certain property. Moreover, they demonstrated that the field $\mathbb{Q}$ does not have this property and produced counterexamples showing that $K_0(L_{\mathbb{Q}}(E))$ and $K_1(L_{\mathbb{Q}}(E))$ do not suffice to classify $L_{\mathbb{Q}}(E)$ in general.

This raises several questions. How can we classify Leavitt path algebras over fields such as $\mathbb{Q}$? What is the proper invariant over other kinds of fields? Do additional K-groups suffice? Can we compute the needed K-groups? In joint work with James Gabe, Efren Ruiz, and Mark Tomforde, we found several answers to these questions.

Given a graph E and a field k, Ara, Brustenga, and Cortiñas have constructed a long exact sequence that relates $K_n(L_{\mathsf{k}}(E))$ to $K_n(\mathsf{k})$ as long as E does not contain any infinite emitters (i.e., vertices that emit infinitely many edges) [6]. The long exact sequence is one of the most useful tools in the computation of algebraic K-groups of Leavitt path algebras, but we cannot directly apply their result because every graph we are considering necessarily has an infinite emitter. In chapter 3, we extend the long exact sequence to include graphs with infinite emitters, enabling us to compute $K_n(L_{\mathsf{k}}(E))$ for several cases (e.g., when the underlying field is finite or when the underlying field is algebraically closed).

One particularly nice class of fields whose K-theory is well understood is the class of number fields. A number field is a finite field extension of $\mathbb{Q}$, and in particular, $\mathbb{Q}$ itself is a number field. In chapter 4, we determine that when k is a number field, the number of singular vertices can be recovered from $K_0(L_{\mathsf{k}}(E))$ and $K_6(L_{\mathsf{k}}(E))$. Hence, if k is a number field, which includes the case that $\mathsf{k} = \mathbb{Q}$, then $L_{\mathsf{k}}(E)$ is classified by $K_0(L_{\mathsf{k}}(E))$ and $K_6(L_{\mathsf{k}}(E))$! In particular, Leavitt path algebras over number fields are the first class known to require a higher K-group for classification.

CHAPTER 2

Preliminaries

We write $\mathbb{N} := \{1, 2, \ldots\}$ for the natural numbers and $\mathbb{Z}^+ := \{0, 1, 2, \ldots\}$ for the non-negative integers. We use the symbol k to denote a field, and we write $\mathsf{k}^\times$ for the multiplicative abelian group $\mathsf{k} \setminus \{0\}$. If R is a ring and $n \in \mathbb{Z}$, we let $K_n(R)$ denote the n^{th} algebraic K-group of R. (We will only be considering algebraic K-theory.)

A *graph* (E^0, E^1, r, s) consists of a set E^0 of vertices, a set E^1 of edges, and maps $r : E^1 \to E^0$ and $s : E^1 \to E^0$ that identify the range and source of each edge. What we call a graph is often referred to as a *directed graph* or a *quiver* in other literature. A graph is *countable* if both the sets E^0 and E^1 are countable, and a graph is *finite* if both the sets E^0 and E^1 are finite.

Assumption: Throughout, graphs are assumed to be at most countable. That is, given a graph E, the set of vertices E^0 is at most countable, and the set of edges E^1 is at most countable.

Let $E = (E^0, E^1, r, s)$ be a graph and let $v \in E^0$ be a vertex of E. We say v is a *sink* if $s^{-1}(v) = \emptyset$, and we say v is an *infinite emitter* if $|s^{-1}(v)| = \infty$. A *singular vertex* is a vertex that is either a sink or an infinite emitter, and we denote the set of singular vertices by E^0_{sing}. We let $E^0_{\text{reg}} := E^0 \setminus E^0_{\text{sing}}$ and refer to an element of E^0_{reg} as a *regular vertex*. Note that a vertex $v \in E^0$ is a *regular vertex* if and only if $0 < |s^{-1}(v)| < \infty$.

If E is a graph, a *path* is a sequence of edges $\alpha := e_1 e_2 \dots e_n$ with $r(e_i) = s(e_{i+1})$ when $n \geq 2$ for $1 \leq i \leq n-1$. The *length* of α is n, and we consider a single edge to be a path of length 1, and a vertex to be a path of length 0. The set of all paths in E is denoted by E^*. A *cycle* is a path $\alpha = e_1 e_2 \dots e_n$ with $n \geq 1$ and $r(e_n) = s(e_1)$.

If $\alpha = e_1 e_2 \dots e_n$ is a cycle, an *exit* for α is an edge $f \in E^1$ such that $s(f) = s(e_i)$ and $f \neq e_i$ for some i. A graph is said to satisfy *Condition (L)* if every cycle in the graph has an exit. An *infinite path* in a graph E is an infinite sequence of edges $\mu := e_1 e_2 \dots$ with $r(e_i) = s(e_{i+1})$ for all $i \in \mathbb{N}$. A graph E is called *cofinal* if whenever $\mu := e_1 e_2 \dots$ is an infinite path in E and $v \in E^0$, then there exists a finite path $\alpha \in E^*$ with $s(\alpha) = v$ and $r(\alpha) = s(e_i)$ for some $i \in \mathbb{N}$.

We say that a graph E is *simple* if E satisfies all of the following three conditions:

(i) E is cofinal,

(ii) E satisfies Condition (L), and

(iii) whenever $v \in E^0$ and $w \in E^0_{\text{sing}}$, there exists a path $\alpha \in E^*$ with $s(\alpha) = v$ and $r(\alpha) = w$.

It is proven in [22, Proposition 4.2] that the following are equivalent:

(1) The graph E is simple.

(2) The Leavitt path algebra $L_{\mathsf{k}}(E)$ is simple for any field k.

(3) The graph C^*-algebra $C^*(E)$ is simple.

Given a graph E, the *vertex matrix* A_E is the $E^0 \times E^0$ matrix whose entries are given by $A_E(v, w) := |\{e \in E^1 : s(e) = v \text{ and } r(e) = w\}|$. We write ∞ for this value when $\{e \in E^1 : s(e) = v \text{ and } r(e) = w\}$ is an infinite set, so A_E takes values in $\mathbb{Z}^+ \cup \{\infty\}$.

Let E be a graph, and let k be a field. We let $(E^1)^*$ denote the set of formal symbols $\{e^* : e \in E^1\}$. The *Leavitt path algebra of E with coefficients in* k, denoted $L_{\mathsf{k}}(E)$, is the free associative k-algebra generated by a set $\{v : v \in E^0\}$ of pairwise orthogonal idempotents, together with a set $\{e, e^* : e \in E^1\}$ of elements, modulo the ideal generated by the following relations:

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3. $e^* f = \delta_{e,f}\, r(e)$ for all $e, f \in E^1$

4. $v = \sum_{\{e \in E^1 : s(e) = v\}} e e^*$ whenever $v \in E^0_{\mathrm{reg}}$.

If $\alpha = e_1 \dots e_n$ is a path of positive length, we define $\alpha^* = e_n^* \dots e_1^*$. One can show that

$$L_{\mathsf{k}}(E) = \mathrm{span}_{\mathsf{k}}\{\alpha\beta^* : \alpha \text{ and } \beta \text{ are paths in } E \text{ with } r(\alpha) = r(\beta)\}.$$

If E is a graph and k is a field, the Leavitt path algebra $L_{\mathsf{k}}(E)$ is unital if and only if the vertex set E^0 is finite, in which case $1 = \sum_{v \in E^0} v$.

We write $e \sim f$ if and only if $e = xy$ and $f = yx$ for some x and y. An idempotent e is *infinite* if there exist orthogonal idempotents f and g such that $e = f + g$ and $e \sim f$ and $g \neq 0$.

A simple ring is called *purely infinite* if every nonzero left ideal contains an infinite idempotent.

It is shown in [4, Theorem 11] that the Leavitt path algebra $L_{\mathsf{k}}(E)$ is purely infinite and simple for any field k if and only if E has the following properties:

(i) E is cofinal,

(ii) E satisfies Condition (L),

(iii) whenever $v \in E^0$ and $w \in E^0_{\mathrm{sing}}$, there exists a path $\alpha \in E^*$ with $s(\alpha) = v$ and $r(\alpha) = w$, and

(iv) whenever $v \in E^0$, there exists a path $\alpha \in E^*$ with $s(\alpha) = v$ such that $r(\alpha)$ is in a cycle.

Suppose E is a graph with a singular vertex $v_0 \in E^0_{\text{sing}}$. We *add a tail* at v_0 by attaching a graph of the form

$$v_0 \longrightarrow v_1 \longrightarrow v_2 \longrightarrow v_3 \longrightarrow \cdots$$

to E at v_0, and in the case v_0 is an infinite emitter, we list the edges of $s^{-1}(v_0)$ as $g_1, g_2, g_3, \ldots$, remove the edges in $s^{-1}(v_0)$, and for each g_j we draw an edge f_j from v_{j-1} to $r(g_j)$. A *desingularization* of E is a graph F obtained by adding a tail at every singular vertex of E. It is shown in [5, Theorem 5.2] that if F is a desingularization of E, then for any field k the Leavitt path algebra $L_{\mathsf{k}}(E)$ is isomorphic to a full corner of the Leavitt path algebra $L_{\mathsf{k}}(F)$, and the algebras $L_{\mathsf{k}}(E)$ and $L_{\mathsf{k}}(F)$ are Morita equivalent.

CHAPTER 3

The Long Exact Sequence of K-groups of Leavitt Path Algebras

The starting point for computing K-groups of a Leavitt path algebra is the long exact sequence of Ara, Brustenga, and Cortiñas. This sequence relates the algebraic K-groups of the Leavitt path algebra to the algebraic K-groups of the underlying field. The theorem requires the associated graph to be row-finite (i.e., all the vertices emit a finite number of edges) but the graphs under our consideration necessarily have have singular vertices. In the first section, we extend the long exact sequence to include arbitrary countable graphs.

With the long exact sequence result thus empowered, we can obtain several computations as long as we know something about the K-theory of the underlying field. We do this in the remaining sections, using these facts from algebraic K-theory:

- For any field k, when $n \leq 0$, we know that $K_n(\mathsf{k}) = 0$. (See section 2.)
- For any field k, we know that $K_0(\mathsf{k}) \cong \mathbb{Z}$. (See section 2.)
- For any field k, we know that $K_1(\mathsf{k}) \cong \mathsf{k}^\times$. (See section 2.)
- If k is a finite field, we know $K_n(\mathsf{k})$ for all n. (See section 3.)
- If k is an algebraically closed field, we know $K_n(\mathsf{k})$ is divisible. (See section 4.)

3.1 Long Exact Sequence for Countable Graphs

In this section we extend the K-theory computation of [6, Theorem 7.6] to Leavitt path algebras of graphs that may contain singular vertices. For an abelian group G and a set S (possibly infinite), we denote the direct sum $\bigoplus_S G$ by G^S. Note that this differs from the notation used in [6] in which the authors denote the direct sum by $G^{(S)}$.

Theorem 3.1.1. *Let $E = (E^0, E^1, r, s)$ be a graph. Decompose the vertices of E as $E^0 = E^0_{\text{reg}} \sqcup E^0_{\text{sing}}$, and with respect to this decomposition write the vertex matrix of E as*

$$\begin{pmatrix} B_E & C_E \\ * & * \end{pmatrix}$$

where B_E and C_E have entries in $\mathbb{Z}^+$ and each $*$ has entries in $\mathbb{Z}^+ \cup \{\infty\}$. If k is a field, then for the Leavitt path algebra $L_{\mathsf{k}}(E)$ there is a long exact sequence

$$\cdots \longrightarrow K_n(\mathsf{k})^{E^0_{\text{reg}}} \xrightarrow{\binom{B_E^t - I}{C_E^t}} K_n(\mathsf{k})^{E^0} \longrightarrow K_n(L_{\mathsf{k}}(E)) \longrightarrow K_{n-1}(\mathsf{k})^{E^0_{\text{reg}}} \xrightarrow{\binom{B_E^t - I}{C_E^t}} \cdots$$

for all $n \in \mathbb{Z}$.

Proof. If E has no singular vertices, then the result holds by [6, Theorem 7.6]. Otherwise, we will apply [6, Theorem 7.6] to a desingularization of E and show that the result holds for E as well. Suppose E has at least one singular vertex. List the singular vertices of E as $E^0_{\text{sing}} := \{v_1^0, v_2^0, v_3^0, \ldots\}$. Note that E^0_{sing} could be finite or countably infinite, but not empty.

Let F be a desingularization of E. In forming F from E, we add a tail to each singular vertex of E and "distribute" the edges of each infinite emitter along the vertices of the tail added to that infinite emitter. For each singular vertex $v_i^0 \in E^0_{\text{sing}}$, let $\{v_i^1, v_i^2, v_i^3, \ldots\}$ denote the vertices of the tail added to v_i^0. (See [10, Section 2] for details.) If A_F is the vertex matrix of F, we will now describe $A_F^t - I$ following [9, Lemma 2.3]. For each $1 \leq i \leq |E^0_{\text{sing}}|$, let D_i denote the $E^0_{\text{sing}} \times \mathbb{N}$ matrix with 1 in the $(i, 1)$ position and zeros elsewhere:

$$D_i = \begin{pmatrix} 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \cdots \\ 0 & 0 & 0 & \cdots \\ 1 & 0 & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Let Z denote the $\mathbb{N} \times \mathbb{N}$ matrix with -1 in each entry of the diagonal and 1 in each entry of the superdiagonal:

$$Z = \begin{pmatrix} -1 & 1 & 0 & 0 & \cdots \\ 0 & -1 & 1 & 0 & \cdots \\ 0 & 0 & -1 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

With respect to the decomposition $E^0_{\text{reg}} \sqcup E^0_{\text{sing}} \sqcup \{v_1^1, v_1^2, v_1^3, \ldots\} \sqcup \{v_2^1, v_2^2, v_2^3, \ldots\} \sqcup \cdots$, we have

$$A_F^t - I = \begin{pmatrix} B_E^t - I & X_1^t & X_2^t & X_3^t & \cdots \\ C_E^t & Y_1^t - I & Y_2^t & Y_3^t & \cdots \\ 0 & D_1^t & Z^t & 0 & \cdots \\ 0 & D_2^t & 0 & Z^t & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

where each X_i^t and each Y_i^t is column-finite.

Since $E^0 \subseteq F^0$, we may define $\iota_n : K_n(\mathsf{k})^{E^0} \to K_n(\mathsf{k})^{F^0}$ and $\iota_n^{\text{reg}} : K_n(\mathsf{k})^{E^0_{\text{reg}}} \to K_n(\mathsf{k})^{F^0}$ to be the inclusion maps. If $\mathbf{x} \in K_n(\mathsf{k})^{E^0_{\text{reg}}}$, then

$$(A_F^t - I)\ \iota_n^{\text{reg}}(\mathbf{x}) = (A_F^t - I) \begin{pmatrix} \mathbf{x} \\ \mathbf{0} \\ \begin{pmatrix} \mathbf{0} \\ \mathbf{0} \\ \vdots \end{pmatrix} \end{pmatrix} = \begin{pmatrix} (B_E^t - I)\mathbf{x} \\ C_E^t \mathbf{x} \\ \begin{pmatrix} \mathbf{0} \\ \mathbf{0} \\ \vdots \end{pmatrix} \end{pmatrix} = \iota_n \left(\begin{pmatrix} B_E^t - I \\ C_E^t \end{pmatrix} \mathbf{x} \right).$$

So, the diagram

$$\begin{array}{ccc} K_n(\mathsf{k})^{E^0_{\mathrm{reg}}} & \xrightarrow{\binom{B_E^t - I}{C_E^t}} & K_n(\mathsf{k})^{E^0} \\ \downarrow{\scriptstyle \iota_n^{\mathrm{reg}}} & & \downarrow{\scriptstyle \iota_n} \\ K_n(\mathsf{k})^{F^0} & \xrightarrow{A_F^t - I} & K_n(\mathsf{k})^{F^0} \end{array} \tag{3.1}$$

commutes for each $n \in \mathbb{Z}$.

By [5, Theorem 5.2], there is an embedding $\phi : L_{\mathsf{k}}(E) \to L_{\mathsf{k}}(F)$ onto a full corner of $L_{\mathsf{k}}(F)$. By [23, Lemma 5.2] ϕ induces an isomorphism $\phi_* : K_n(L_{\mathsf{k}}(E)) \to K_n(L_{\mathsf{k}}(F))$ for each $n \in \mathbb{Z}$. Since F has no singular vertices, [6, Theorem 7.6] implies there exists a long exact sequence

$$\cdots \longrightarrow K_n(\mathsf{k})^{F^0} \xrightarrow{A_F^t - I} K_n(\mathsf{k})^{F^0} \xrightarrow{f} K_n(L_{\mathsf{k}}(F)) \xrightarrow{g} K_{n-1}(\mathsf{k})^{F^0} \xrightarrow{A_F^t - I} \cdots \tag{3.2}$$

Combining (3.2) with (3.1) and the isomorphisms $\phi_* : K_n(L_{\mathsf{k}}(E)) \to K_n(L_{\mathsf{k}}(F))$ we obtain a commutative diagram

$$\begin{array}{ccccccccc} \cdots & & K_n(\mathsf{k})^{E^0_{\mathrm{reg}}} & \xrightarrow{\binom{B_E^t - I}{C_E^t}} & K_n(\mathsf{k})^{E^0} & & K_n(L_{\mathsf{k}}(E)) & & K_{n-1}(\mathsf{k})^{E^0_{\mathrm{reg}}} \xrightarrow{\binom{B_E^t - I}{C_E^t}} \cdots \\ & & \downarrow{\scriptstyle \iota_n^{\mathrm{reg}}} & & \downarrow{\scriptstyle \iota_n} & & \downarrow{\scriptstyle \phi_*} & & \downarrow{\scriptstyle \iota_{n-1}^{\mathrm{reg}}} \\ \cdots & \longrightarrow K_n(\mathsf{k})^{F^0} & \xrightarrow{A_F^t - I} & K_n(\mathsf{k})^{F^0} & \xrightarrow{f} & K_n(L_{\mathsf{k}}(F)) & \xrightarrow{g} & K_{n-1}(\mathsf{k})^{F^0} & \xrightarrow{A_F^t - I} \cdots \end{array}$$

with the lower row exact.

Define $f_0 : K_n(\mathsf{k})^{E^0} \to K_n(L_{\mathsf{k}}(E))$ by $f_0 := \phi_*^{-1} \circ f \circ \iota_n$. Define $g_0 : K_n(L_{\mathsf{k}}(E)) \to$

$K_{n-1}(\mathsf{k})^{E^0_{\mathrm{reg}}}$ by $g_0 := \pi^{\mathrm{reg}}_{n-1} \circ g \circ \phi_*$, where $\pi^{\mathrm{reg}}_n : K_n(\mathsf{k})^{F^0} \to K_n(\mathsf{k})^{E^0_{\mathrm{reg}}}$ by

$$\pi^{\mathrm{reg}}_n \begin{pmatrix} \mathbf{x} \\ \mathbf{y} \\ \begin{pmatrix} \mathbf{z}_1 \\ \mathbf{z}_2 \\ \vdots \end{pmatrix} \end{pmatrix} = \mathbf{x}.$$

Altogether, we have the diagram

$$\begin{array}{ccccccccc} \cdots \longrightarrow K_n(\mathsf{k})^{E^0_{\mathrm{reg}}} & \xrightarrow{\binom{B_E^t - I}{C_E^t}} & K_n(\mathsf{k})^{E^0} & \xrightarrow{f_0} & K_n(L_{\mathsf{k}}(E)) & \xrightarrow{g_0} & K_{n-1}(\mathsf{k})^{E^0_{\mathrm{reg}}} & \xrightarrow{\binom{B_E^t - I}{C_E^t}} & \cdots \\ \downarrow{\scriptstyle \iota_n^{\mathrm{reg}}} & & \downarrow{\scriptstyle \iota_n} & & \downarrow{\scriptstyle \phi_*} & & \downarrow{\scriptstyle \iota_{n-1}^{\mathrm{reg}}} & & \\ \cdots \longrightarrow K_n(\mathsf{k})^{F^0} & \xrightarrow{A_F^t - I} & K_n(\mathsf{k})^{F^0} & \xrightarrow{f} & K_n(L_{\mathsf{k}}(F)) & \xrightarrow{g} & K_{n-1}(\mathsf{k})^{F^0} & \xrightarrow{A_F^t - I} & \cdots \end{array} \tag{3.3}$$

and will show that this diagram commutes and that the upper row is exact. To do so, we will frequently need:

$$\operatorname{im} g \subseteq \operatorname{im} \iota^{\mathrm{reg}}_{n-1}, \tag{3.4}$$

so we prove this now. Let $(\mathbf{u}, \mathbf{v}, (\mathbf{w}_1, \mathbf{w}_2, \ldots))^t \in \ker(A_F^t - I) = \operatorname{im} g \subseteq K_{n-1}(\mathsf{k})^{F^0}$ written with respect to the decomposition $F^0 = E^0_{\mathrm{reg}} \sqcup E^0_{\mathrm{sing}} \sqcup \{v_1^1, v_1^2, v_1^3, \ldots\} \sqcup \{v_2^1, v_2^2, v_2^3, \ldots\} \sqcup \cdots$. Then

$$\begin{pmatrix} \mathbf{0} \\ \mathbf{0} \\ \begin{pmatrix} \mathbf{0} \\ \mathbf{0} \\ \vdots \end{pmatrix} \end{pmatrix} = (A_F^t - I) \begin{pmatrix} \mathbf{u} \\ \mathbf{v} \\ \begin{pmatrix} \mathbf{w}_1 \\ \mathbf{w}_2 \\ \vdots \end{pmatrix} \end{pmatrix} = \begin{pmatrix} (B_E^t - I)\mathbf{u} + X_1^t \mathbf{v} + X_2^t \mathbf{w}_1 + X_3^t \mathbf{w}_2 + \cdots \\ C_E^t \mathbf{u} + (Y_1^t - I)\mathbf{v} + Y_2^t \mathbf{w}_1 + Y_3^t \mathbf{w}_2 + \cdots \\ D_1^t \mathbf{v} + Z^t \mathbf{w}_1 \\ D_2^t \mathbf{v} + Z^t \mathbf{w}_2 \\ D_3^t \mathbf{v} + Z^t \mathbf{w}_3 \\ \vdots \end{pmatrix}.$$

We have $D_i^t\mathbf{v} + Z\mathbf{w}_i = 0$ for all $1 \leq i \leq |E^0_{\text{sing}}|$. For a fixed $i \in \mathbb{N}$, define $\mathbf{v} := (v_1, v_2, \ldots)$ and $\mathbf{w}_i := (w_{i_1}, w_{i_2}, \ldots)$. Then $D_i^t\mathbf{v} + Z^t\mathbf{w}_i = 0$ implies

$$\begin{pmatrix} v_i \\ 0 \\ 0 \\ \vdots \end{pmatrix} + \begin{pmatrix} -w_{i_1} \\ w_{i_1} - w_{i_2} \\ w_{i_2} - w_{i_3} \\ \vdots \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \end{pmatrix},$$

and we conclude that $v_i = w_{i_1} = w_{i_2} = w_{i_3} = \cdots$ for each $1 \leq i \leq |E^0_{\text{sing}}|$. Since $\mathbf{w}_i$ is in the direct sum $K_{n-1}(\mathsf{k})^{\{v_i^1, v_i^2, \ldots\}}$, the entries w_{i_k} are eventually 0 for each i, so $\mathbf{0} = \mathbf{v} = \mathbf{w}_1 = \mathbf{w}_2 = \cdots$. This implies $\operatorname{im} g = \ker(A_F^t - I) \subseteq \operatorname{im} \iota_n^{\text{reg}}$.

Now we check commutativity of (3.3). We already have that $(A_F^t - I) \circ \iota_n^{\text{reg}} = \iota_n \circ \begin{pmatrix} B_E^t - I \\ C_E^t \end{pmatrix}$ from (3.1). From the definition of f_0 we have $\phi_* \circ f_0 = \phi_* \circ \phi_*^{-1} \circ f \circ \iota_n = f \circ \iota_n$. Finally, $\iota_{n-1}^{\text{reg}} \circ g_0 = \iota_{n-1}^{\text{reg}} \circ \pi_{n-1}^{\text{reg}} \circ g \circ \phi_* = g \circ \phi_*$ from the definition of g_0 and by (3.4). Hence (3.3) is commutative.

Next, we verify exactness at $K_n(\mathsf{k})^{E^0}$, $K_n(L_\mathsf{k}(E))$, and $K_{n-1}(\mathsf{k})^{E^0}$.

Step 1: $\operatorname{im} \begin{pmatrix} B_E^t - I \\ C_E^t \end{pmatrix} = \ker f_0$. Because of the commutativity and exactness of (3.2):

$$f_0 \circ \begin{pmatrix} B_E^t - I \\ C_E^t \end{pmatrix}(\mathbf{x}) = \phi_*^{-1} \circ f \circ \iota_n \circ \begin{pmatrix} B_E^t - I \\ C_E^t \end{pmatrix}(\mathbf{x}) = \phi_*^{-1} \circ f \circ (A_F^t - I) \circ \iota_n^{\text{reg}}(\mathbf{x}) = 0$$

for all $\mathbf{x} \in K_n(\mathsf{k})^{E^0_{\text{reg}}}$. Hence $\operatorname{im} \begin{pmatrix} B_E^t - I \\ C_E^t \end{pmatrix} \subseteq \ker f_0$. For the reverse inclusion, if $(\mathbf{x}, \mathbf{y})^t \in \ker f_0$, then $\phi_*^{-1} \circ f \circ \iota_n((\mathbf{x}, \mathbf{y})^t) = 0$, and since ϕ_* is an isomorphism, $\iota_n((\mathbf{x}, \mathbf{y})^t) \in \ker f = \operatorname{im}(A_F^t - I)$. Let $(\mathbf{u}, \mathbf{v}, (\mathbf{w}_1, \mathbf{w}_2, \ldots))^t \in K_n(\mathsf{k})^{F^0}$ be an element that $A_F^t - I$ maps to $\iota_n((\mathbf{x}, \mathbf{y})^t)$, written with respect to the decomposition $F^0 =$

$E^0_{\mathrm{reg}} \sqcup E^0_{\mathrm{sing}} \sqcup \{v_1^1, v_1^2, v_1^3, \ldots\} \sqcup \{v_2^1, v_2^2, v_2^3, \ldots\} \sqcup \cdots$. Then

$$\begin{pmatrix} \mathbf{x} \\ \mathbf{y} \\ \begin{pmatrix} \mathbf{0} \\ \mathbf{0} \\ \vdots \end{pmatrix} \end{pmatrix} = \iota_n \begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix} = (A_F^t - I) \begin{pmatrix} \mathbf{u} \\ \mathbf{v} \\ \begin{pmatrix} \mathbf{w}_1 \\ \mathbf{w}_2 \\ \vdots \end{pmatrix} \end{pmatrix} = \begin{pmatrix} (B_E^t - I)\mathbf{u} + X_1^t\mathbf{v} + X_2^t\mathbf{w}_1 + \cdots \\ C_E^t\mathbf{u} + (Y_1^t - I)\mathbf{v} + Y_2^t\mathbf{w}_1 + \cdots \\ D_1^t\mathbf{v} + Z^t\mathbf{w}_1 \\ D_2^t\mathbf{v} + Z^t\mathbf{w}_2 \\ D_3^t\mathbf{v} + Z^t\mathbf{w}_3 \\ \vdots \end{pmatrix}.$$

Using the computations following (3.4), we obtain $\mathbf{0} = \mathbf{v} = \mathbf{w}_1 = \mathbf{w}_2 = \cdots$ and so

$$\begin{pmatrix} \mathbf{x} \\ \mathbf{y} \\ \begin{pmatrix} \mathbf{0} \\ \mathbf{0} \\ \vdots \end{pmatrix} \end{pmatrix} = \begin{pmatrix} (B_E^t - I)\mathbf{u} \\ C_E^t\mathbf{u} \\ \begin{pmatrix} \mathbf{0} \\ \mathbf{0} \\ \vdots \end{pmatrix} \end{pmatrix}.$$

Thus, $(\mathbf{x}, \mathbf{y})^t = \begin{pmatrix} B_E^t - I \\ C_E^t \end{pmatrix} \mathbf{u}$, so $(\mathbf{x}, \mathbf{y})^t \in \operatorname{im} \begin{pmatrix} B_E^t - I \\ C_E^t \end{pmatrix}$ and $\ker f_0 \subseteq \operatorname{im} \begin{pmatrix} B_E^t - I \\ C_E^t \end{pmatrix}$.

Step 2: $\operatorname{im} f_0 = \ker g_0$. Since $\operatorname{im} f = \ker g$,

$$g_0 \circ f_0 = \pi^{\mathrm{reg}}_{n-1} \circ g \circ \phi_* \circ \phi_*^{-1} \circ f \circ \iota_n = \pi^{\mathrm{reg}}_{n-1} \circ g \circ f \circ \iota_n = 0,$$

implying that $\operatorname{im} f_0 \subseteq \ker g_0$. For the reverse inclusion, let $x \in \ker g_0$. Then $\mathbf{0} =$

$g_0(x) = \pi^{\text{reg}}_{n-1} \circ g \circ \phi_*(x)$, and by (3.4) this implies $\phi_*(x) \in \ker g = \operatorname{im} f$. Let

$$\begin{pmatrix} \mathbf{p} \\ \mathbf{q} \\ \begin{pmatrix} \mathbf{r}_1 \\ \mathbf{r}_2 \\ \vdots \end{pmatrix} \end{pmatrix} \in K_n(\mathsf{k})^{F^0}$$

be an element that f maps to $\phi_*(x)$, written with respect to the decomposition $F^0 = E^0_{\text{reg}} \sqcup E^0_{\text{sing}} \sqcup \{v_1^1, v_1^2, v_1^3, \ldots\} \sqcup \{v_2^1, v_2^2, v_2^3, \ldots\} \sqcup \cdots$. For each $1 \leq i \leq |E^0_{\text{sing}}|$, write

$$\mathbf{r}_i = \begin{pmatrix} r_{i_1} \\ r_{i_2} \\ r_{i_3} \\ \vdots \end{pmatrix}$$

and define

$$b_i := \sum_{j=1}^{\infty} r_{i_j} \qquad \text{and} \qquad c_{i_k} := \sum_{j=k+1}^{\infty} r_{i_j}.$$

Since $\mathbf{r}_i$ is in the direct sum $K_n(\mathsf{k})^{\{v_i^1, v_i^2, \ldots\}}$ for each i, each of the above sums has only finitely many nonzero terms. Also, since $(\mathbf{r}_1, \mathbf{r}_2, \ldots)^t$ is in the direct sum $K_n(\mathsf{k})^{F^0 \setminus E^0}$, we have that $\mathbf{r}_i = 0$ eventually, so

$$\mathbf{b} := \begin{pmatrix} b_1 \\ b_2 \\ \vdots \end{pmatrix} \in K_n(\mathsf{k})^{E^0_{\text{sing}}} \quad \text{and} \quad \mathbf{c} := \begin{pmatrix} \mathbf{c}_1 \\ \mathbf{c}_2 \\ \vdots \end{pmatrix} \in K_n(\mathsf{k})^{F^0 \setminus E^0} \text{ where } \mathbf{c}_i := \begin{pmatrix} c_{i_1} \\ c_{i_2} \\ \vdots \end{pmatrix}.$$

Now set

$$\mathbf{u} := -X_1^t \mathbf{b} - X_2^t \mathbf{c}_1 - X_3^t \mathbf{c}_2 - \cdots$$

and

$$\mathbf{v} := -(Y_1^t - I)\mathbf{b} - Y_2^t\mathbf{c}_1 - Y_3^t\mathbf{c}_2 - \cdots,$$

which are finite sums since $(\mathbf{c}_1, \mathbf{c}_2, \ldots)^t$ is in the direct sum. This gives

$$(A_F^t - I)\begin{pmatrix} \mathbf{0} \\ \mathbf{b} \\ \begin{pmatrix} \mathbf{c}_1 \\ \mathbf{c}_2 \\ \vdots \end{pmatrix} \end{pmatrix} = \begin{pmatrix} -\mathbf{u} \\ -\mathbf{v} \\ \begin{pmatrix} \mathbf{r}_1 \\ \mathbf{r}_2 \\ \vdots \end{pmatrix} \end{pmatrix}.$$

Thus,

$$\begin{pmatrix} -\mathbf{u} \\ -\mathbf{v} \\ \begin{pmatrix} \mathbf{r}_1 \\ \mathbf{r}_2 \\ \vdots \end{pmatrix} \end{pmatrix} \in \operatorname{im}(A_F^t - I) = \ker f,$$

and

$$f_0\begin{pmatrix} \mathbf{p}+\mathbf{u} \\ \mathbf{q}+\mathbf{v} \end{pmatrix} = \phi_*^{-1} \circ f \circ \iota_n \begin{pmatrix} \mathbf{p}+\mathbf{u} \\ \mathbf{q}+\mathbf{v} \end{pmatrix} = \phi_*^{-1} \circ f \begin{pmatrix} \mathbf{p}+\mathbf{u} \\ \mathbf{q}+\mathbf{v} \\ \begin{pmatrix} \mathbf{0} \\ \mathbf{0} \\ \vdots \end{pmatrix} \end{pmatrix} = \phi_*^{-1} \circ f \begin{pmatrix} \mathbf{p} \\ \mathbf{q} \\ \begin{pmatrix} \mathbf{r}_1 \\ \mathbf{r}_2 \\ \vdots \end{pmatrix} \end{pmatrix} = x,$$

so $x \in \operatorname{im} f_0$.

Step 3: $\operatorname{im} g_0 = \ker \begin{pmatrix} B_E^t - I \\ C_E^t \end{pmatrix}$. First we will show $\begin{pmatrix} B_E^t - I \\ C_E^t \end{pmatrix} \circ\ g_0 = 0$, which happens if

and only if $\iota_n \circ \begin{pmatrix} B_E^t - I \\ C_E^t \end{pmatrix} \circ g_0 = 0$ because ι_n is injective. Since

$$\begin{aligned}
\iota_n \circ \begin{pmatrix} B_E^t - I \\ C_E^t \end{pmatrix} \circ g_0 &= (A_F^t - I) \circ \iota_n^{\mathrm{reg}} \circ g_0 && \text{(by commutativity of (3.1))} \\
&= (A_F^t - I) \circ \iota_n^{\mathrm{reg}} \circ \pi_{n-1}^{\mathrm{reg}} \circ g \circ \phi_* && \text{(by definition of } g_0) \\
&= (A_F^t - I) \circ g \circ \phi_* && \text{(by (3.4))} \\
&= 0 \circ \phi_* = 0 && \text{(by exactness of (3.2)),}
\end{aligned}$$

it follows $\operatorname{im} g_0 \subseteq \ker \begin{pmatrix} B_E^t - I \\ C_E^t \end{pmatrix}$. For the reverse inclusion, let $\mathbf{u} \in \ker \begin{pmatrix} B_E^t - I \\ C_E^t \end{pmatrix}$. By the commutativity of (3.1) and exactness of (3.2), we have $\iota_{n-1}^{\mathrm{reg}}(\mathbf{u}) \in \ker(A_F^t - I) = \operatorname{im} g$. Let $a \in K_n(L_{\mathsf{k}}(F))$ be an element that g maps to $\iota_{n-1}^{\mathrm{reg}}(\mathbf{u})$. Then

$$g_0(\phi_*^{-1}(a)) = \pi_{n-1}^{\mathrm{reg}} \circ g \circ \phi_*(\phi_*^{-1}(a)) = \pi_{n-1}^{\mathrm{reg}} \circ g(a) = \pi_{n-1}^{\mathrm{reg}} \circ \iota_{n-1}^{\mathrm{reg}}(\mathbf{u}) = \mathbf{u},$$

so $\mathbf{u} \in \operatorname{im} g_0$. Thus $\ker \begin{pmatrix} B_E^t - I \\ C_E^t \end{pmatrix} \subseteq \operatorname{im} g_0$, and (3.3) is exact. □

3.2 Computing K_n for $n \leq 1$

The following lemma regarding negative K-theory is included for the convenience of the reader. The proof requires putting together a few results in [21].

Lemma 3.2.1. *If R is a left regular ring and $n \geq 1$, then $K_{-n}(R) = \{0\}$. In particular, if k is a field and $n \geq 1$, then $K_{-n}(\mathsf{k}) = \{0\}$.*

Proof. The lemma is stated as fact in [21, Definition 3.3.1], referring to [21, Corollary 3.2.20] and [21, Theorem 3.2.3], but [21, Corollary 3.2.13] is also needed. For the last claim, note that a field is trivially a left regular ring. □

Combining Theorem 3.1.1 with facts about the algebraic K-theory of fields, we obtain the following proposition, which is well known in some special cases (e.g., [22, Proposition 5.1]).

Proposition 3.2.2. *Let E be a graph, let k be a field, and consider the Leavitt path algebra $L_{\mathsf{k}}(E)$.*

(i) If $n \leq -1$, then $K_n(L_{\mathsf{k}}(E)) = \{0\}$.

(ii) $$K_0(L_{\mathsf{k}}(E)) \cong \operatorname{coker}\left(\begin{pmatrix} B_E^t - I \\ C_E^t \end{pmatrix} : \mathbb{Z}^{E^0_{\mathrm{reg}}} \to \mathbb{Z}^{E^0}\right).$$

(iii) $K_1(L_{\mathsf{k}}(E))$ is isomorphic to a direct sum:

$$K_1(L_{\mathsf{k}}(E)) \cong \operatorname{coker}\left(\begin{pmatrix} B_E^t - I \\ C_E^t \end{pmatrix} : (\mathsf{k}^\times)^{E^0_{\mathrm{reg}}} \to (\mathsf{k}^\times)^{E^0}\right) \oplus \ker\left(\begin{pmatrix} B_E^t - I \\ C_E^t \end{pmatrix} : \mathbb{Z}^{E^0_{\mathrm{reg}}} \to \mathbb{Z}^{E^0}\right).$$

Proof. By Lemma 3.2.1, if k is any field and $n \leq -1$, then $K_n(\mathsf{k}) = \{0\}$. Combining this with Theorem 3.1.1, item (i) follows with an exactness argument. By [21, Example 1.1.6], if k is any field, then $K_0(\mathsf{k}) = \mathbb{Z}$. Using Theorem 3.1.1 and the first isomorphism theorem, item (ii) follows with an exactness argument.

By [27, Example III.1.1.2] or by [21, Proposition 2.2.2], if k is any field, then $K_1(\mathsf{k}) = \mathsf{k}^\times$. So at position $n = 1$, the exact sequence of Theorem 3.1.1 takes the form

$$\cdots \longrightarrow (\mathsf{k}^\times)^{E^0_{\mathrm{reg}}} \xrightarrow{\binom{B_E^t - I}{C_E^t}} (\mathsf{k}^\times)^{E^0} \longrightarrow K_1(L_{\mathsf{k}}(E)) \longrightarrow \mathbb{Z}^{E^0_{\mathrm{reg}}} \xrightarrow{\binom{B_E^t - I}{C_E^t}} \cdots,$$

which induces the short exact sequence

$$0 \longrightarrow \operatorname{coker} \begin{pmatrix} B_E^t - I \\ C_E^t \end{pmatrix} \longrightarrow K_1(L_{\mathsf{k}}(E)) \longrightarrow \ker \begin{pmatrix} B_E^t - I \\ C_E^t \end{pmatrix} \longrightarrow 0 \, .$$

Since $\ker \left(\begin{smallmatrix} B_E^t - I \\ C_E^t \end{smallmatrix} \right)$ is a subgroup of the free abelian group $\mathbb{Z}^{E^0_{\mathrm{reg}}}$, it follows that $\ker \left(\begin{smallmatrix} B_E^t - I \\ C_E^t \end{smallmatrix} \right)$ is a free abelian group. Hence the short exact sequence splits and item (iii) holds. □

Remark 3.2.3. When we write $K_1(\mathsf{k})$ additively, the matrix multiplication

$$\begin{pmatrix} B_E^t - I \\ C_E^t \end{pmatrix} : K_1(\mathsf{k})^{E^0_{\mathrm{reg}}} \to K_1(\mathsf{k})^{E^0}$$

is the usual matrix multiplication. When we identify $K_1(\mathsf{k})$ with $\mathsf{k}^\times$, we write

$$\begin{pmatrix} B_E^t - I \\ C_E^t \end{pmatrix} : (\mathsf{k}^\times)^{E^0_{\mathrm{reg}}} \to (\mathsf{k}^\times)^{E^0}$$

and in this case the group operation is written multiplicatively, not additively. For example, if we have

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} : (\mathsf{k}^\times)^2 \to (\mathsf{k}^\times)^2$$

then this map takes

$$\begin{pmatrix} x \\ y \end{pmatrix} \in (\mathsf{k}^\times)^2 \quad \text{to} \quad \begin{pmatrix} x^a y^b \\ x^c y^d \end{pmatrix} \in (\mathsf{k}^\times)^2.$$

Theorem 3.2.4. *Let* k *be a field and* $n \in \mathbb{N}$. *If* E *is a graph with only finitely many vertices, then there exist* $d_1, \ldots, d_k \in \{2, 3, 4, \ldots\}$ *and* $m \in \{0, 1, 2, \ldots\}$ *such that*

$d_i | d_{i+1}$ *for* $1 \leq i \leq k-1$ *and*

$$\operatorname{coker}\left(\begin{pmatrix} B_E^t - I \\ C_E^t \end{pmatrix} : K_n(\mathsf{k})^{E^0_{\mathrm{reg}}} \to K_n(\mathsf{k})^{E^0}\right)$$
$$\cong K_n(\mathsf{k}) \Big/ \langle d_1 x : x \in K_n(\mathsf{k}) \rangle \oplus \cdots \oplus K_n(\mathsf{k}) \Big/ \langle d_k x : x \in K_n(\mathsf{k}) \rangle \oplus K_n(\mathsf{k})^{m+|E^0_{\mathrm{sing}}|}$$

and

$$\ker\left(\begin{pmatrix} B_E^t - I \\ C_E^t \end{pmatrix} : K_n(\mathsf{k})^{E^0_{\mathrm{reg}}} \to K_n(\mathsf{k})^{E^0}\right)$$
$$\cong K_n(\mathsf{k})^m \oplus \left(\bigoplus_{i=1}^{k} \ker((d_i) : K_n(\mathsf{k}) \to K_n(\mathsf{k}))\right).$$

Moreover,

$$K_0(L_{\mathsf{k}}(E)) \cong \mathbb{Z}_{d_1} \oplus \cdots \oplus \mathbb{Z}_{d_k} \oplus \mathbb{Z}^{m+|E^0_{\mathrm{sing}}|}$$

and

$$K_1(L_{\mathsf{k}}(E)) \cong \mathsf{k}^\times \Big/ \{x^{d_1} : x \in \mathsf{k}^\times\} \oplus \cdots \oplus \mathsf{k}^\times \Big/ \{x^{d_k} : x \in \mathsf{k}^\times\} \oplus (\mathsf{k}^\times)^{m+|E^0_{\mathrm{sing}}|} \oplus \mathbb{Z}^m.$$

Proof. If $|E^0| < \infty$, then the matrix $\begin{pmatrix} B_E^t - I \\ C_E^t \end{pmatrix}$ has Smith normal form $\begin{pmatrix} D \\ 0 \end{pmatrix}$, where 0 is the $|E^0_{\mathrm{sing}}| \times |E^0_{\mathrm{reg}}|$ matrix with 0 in each entry and D is the $|E^0_{\mathrm{reg}}| \times |E^0_{\mathrm{reg}}|$ diagonal matrix $\operatorname{diag}(1, \ldots, 1, d_1, \ldots, d_k, 0, \ldots, 0)$, with 0 in each of the last m entries for some $m \in \{0, 1, 2, \ldots\}$. If k is a field, then

$$\operatorname{coker}\left(\begin{pmatrix} B_E^t - I \\ C_E^t \end{pmatrix} : K_n(\mathsf{k})^{E^0_{\mathrm{reg}}} \to K_n(\mathsf{k})^{E^0}\right) \cong \operatorname{coker}\left(\begin{pmatrix} D \\ 0 \end{pmatrix} : K_n(\mathsf{k})^{E^0_{\mathrm{reg}}} \to K_n(\mathsf{k})^{E^0}\right)$$
$$\cong K_n(\mathsf{k}) \Big/ \langle d_1 x : x \in K_n(\mathsf{k}) \rangle \oplus \cdots \oplus K_n(\mathsf{k}) \Big/ \langle d_k x : x \in K_n(\mathsf{k}) \rangle \oplus K_n(\mathsf{k})^{m+|E^0_{\mathrm{sing}}|}$$

and

$$\ker\left(\begin{pmatrix} B_E^t - I \\ C_E^t \end{pmatrix} : K_n(\mathsf{k})^{E^0_{\mathrm{reg}}} \to K_n(\mathsf{k})^{E^0}\right) \cong \ker\left(\begin{pmatrix} D \\ 0 \end{pmatrix} : K_n(\mathsf{k})^{E^0_{\mathrm{reg}}} \to K_n(\mathsf{k})^{E^0}\right)$$
$$\cong K_n(\mathsf{k})^m \oplus \bigoplus_{i=1}^{k} \ker\left((d_i) : K_n(\mathsf{k}) \to K_n(\mathsf{k})\right).$$

When $n = 0$ or $n = 1$, then the last claim follows from Proposition 3.2.2 by substituting $\left(\begin{smallmatrix} D \\ 0 \end{smallmatrix}\right)$ for $\left(\begin{smallmatrix} B_E^t - I \\ C_E^t \end{smallmatrix}\right)$ in Proposition 3.2.2 (ii) and (iii). □

Remark 3.2.5. In Theorem 3.2.4, the case when $k = 0$ (and the list $d_1, \ldots, d_k$ is empty) is possible.

3.3 K-theory for Leavitt Path Algebras over Finite Fields

In this section we compute the K-groups of a Leavitt path algebra over a finite field $\mathsf{k} = \mathbb{F}_q$ with q elements. (So $q = p^k$ for some prime p, where p is the characteristic of the field.)

Proposition 3.3.1. *Let E be a graph, let $\mathbb{F}_q$ be a finite field with q elements, and consider the Leavitt path algebra $L_{\mathbb{F}_q}(E)$. If $n \geq 2$ is even and $n = 2j$ for some $j \in \mathbb{N}$, then*

$$K_n(L_{\mathbb{F}_q}(E)) \cong \ker\left(\begin{pmatrix} B_E^t - I \\ C_E^t \end{pmatrix} : \mathbb{Z}_{q^j-1}^{E^0_{\mathrm{reg}}} \to \mathbb{Z}_{q^j-1}^{E^0}\right).$$

Proof. By [20, Theorem 8(i)], if $j \geq 1$, then $K_{2j}(\mathbb{F}_q) = \{0\}$, and if $j \geq 2$, then $K_{2j-1}(\mathbb{F}_q) = \mathbb{Z}_{q^j-1}$. Hence the exact sequence of Theorem 3.1.1 takes the form

$$\cdots \xrightarrow{\binom{B_E^t - I}{C_E^t}} \{0\}^{E^0} \xrightarrow{\phi} K_n(L_{\mathbb{F}_q}(E)) \xrightarrow{\psi} \mathbb{Z}_{q^j-1}^{E^0_{\mathrm{reg}}} \xrightarrow{\binom{B_E^t - I}{C_E^t}} \mathbb{Z}_{q^j-1}^{E^0} \longrightarrow \cdots,$$

where $n = 2j$. By exactness, ψ is injective and

$$K_n(L_{\mathbb{F}_q}(E)) \cong \operatorname{im}\psi = \ker \begin{pmatrix} B_E^t - I \\ C_E^t \end{pmatrix}.$$

□

Proposition 3.3.2. *Let E be a graph, let $\mathbb{F}_q$ be a finite field with q elements, and consider the Leavitt path algebra $L_{\mathbb{F}_q}(E)$. If $n \geq 3$ is odd and $n = 2j - 1$ for some $j \in \mathbb{N}$, then*

$$K_n(L_{\mathbb{F}_q}(E)) \cong \operatorname{coker}\left(\begin{pmatrix} B_E^t - I \\ C_E^t \end{pmatrix} : \mathbb{Z}_{q^j-1}^{E^0_{\mathrm{reg}}} \to \mathbb{Z}_{q^j-1}^{E^0} \right).$$

Proof. By [20, Theorem 8(i)], if $j \geq 1$, then $K_{2j}(\mathbb{F}_q) = \{0\}$, and if $j \geq 2$, then $K_{2j-1}(\mathbb{F}_q) = \mathbb{Z}_{q^j-1}$. So the exact sequence of Theorem 3.1.1 becomes

$$\cdots \longrightarrow \mathbb{Z}_{q^j-1}^{E^0_{\mathrm{reg}}} \xrightarrow{\binom{B_E^t - I}{C_E^t}} \mathbb{Z}_{q^j-1}^{E^0} \xrightarrow{\phi} K_n(L_{\mathbb{F}_q}(E)) \xrightarrow{\psi} \{0\}^{E^0_{\mathrm{reg}}} \longrightarrow \cdots,$$

where $n = 2j - 1$. By exactness, ϕ is surjective and the first isomorphism theorem implies

$$K_n(L_{\mathbb{F}_q}(E)) = \operatorname{im}\phi \cong \mathbb{Z}_{q^j-1}^{E^0_{\mathrm{reg}}} / \ker\phi = \mathbb{Z}_{q^j-1}^{E^0_{\mathrm{reg}}} / \operatorname{im}\begin{pmatrix} B_E^t - I \\ C_E^t \end{pmatrix} = \operatorname{coker}\begin{pmatrix} B_E^t - I \\ C_E^t \end{pmatrix}.$$

□

Theorem 3.3.3. *Let E be a graph, let $\mathbb{F}_q$ be a finite field with q elements, and consider the Leavitt path algebra $L_{\mathbb{F}_q}(E)$. Then for any $n \in \mathbb{Z}$ we have*

$$K_n(L_{\mathbb{F}_q}(E)) \cong \begin{cases} 0 & \text{if } n \leq -1 \\ \operatorname{coker}\left(\begin{pmatrix} B_E^t - I \\ C_E^t \end{pmatrix} : \mathbb{Z}^{E^0_{\mathrm{reg}}} \to \mathbb{Z}^{E^0}\right) & \text{if } n = 0 \\ \operatorname{coker}\left(\begin{pmatrix} B_E^t - I \\ C_E^t \end{pmatrix} : \mathbb{Z}^{E^0_{\mathrm{reg}}}_{q-1} \to \mathbb{Z}^{E^0}_{q-1}\right) & \\ \quad \oplus \ker\left(\begin{pmatrix} B_E^t - I \\ C_E^t \end{pmatrix} : \mathbb{Z}^{E^0_{\mathrm{reg}}} \to \mathbb{Z}^{E^0}\right) & \text{if } n = 1 \\ \ker\left(\begin{pmatrix} B_E^t - I \\ C_E^t \end{pmatrix} : \mathbb{Z}^{E^0_{\mathrm{reg}}}_{q^j-1} \to \mathbb{Z}^{E^0}_{q^j-1}\right) & \text{if } n \geq 2 \text{ is even and } n = 2j \\ \operatorname{coker}\left(\begin{pmatrix} B_E^t - I \\ C_E^t \end{pmatrix} : \mathbb{Z}^{E^0_{\mathrm{reg}}}_{q^j-1} \to \mathbb{Z}^{E^0}_{q^j-1}\right) & \text{if } n \geq 3 \text{ is odd and } n = 2j-1. \end{cases}$$

Proof. The case when $n \geq 2$ is even follows from Proposition 3.3.1, and the case when $n \geq 3$ is odd follows from Proposition 3.3.2. The case when $n \leq -1$ follows from Proposition 3.2.2(i), and the case when $n = 0$ follows from Proposition 3.2.2(ii). When $n = 1$, Proposition 3.2.2(iii) shows that

$$K_1(L_{\mathsf{k}}(E)) \cong \operatorname{coker}\left(\begin{pmatrix} B_E^t - I \\ C_E^t \end{pmatrix} : (\mathbb{F}_q^\times)^{E^0_{\mathrm{reg}}} \to (\mathbb{F}_q^\times)^{E^0}\right) \oplus \ker\left(\begin{pmatrix} B_E^t - I \\ C_E^t \end{pmatrix} : \mathbb{Z}^{E^0_{\mathrm{reg}}} \to \mathbb{Z}^{E^0}\right).$$

Since $\mathbb{F}_q$ is a finite field with q elements, if follows that the multiplicative group $\mathbb{F}_q^\times$ is a cyclic group of order $q-1$ (see, for example, [15, Theorem 5.3] or [12, Theorem 22.2]). Thus the multiplicative group $\mathbb{F}_q^\times$ is isomorphic to the additive group $\mathbb{Z}_{q-1}$, and there is an element $\alpha \in \mathbb{F}_q^\times$ of multiplicative order $q-1$ with the isomorphism from $\mathbb{F}_q^\times$ to $\mathbb{Z}_{q-1}$ given by $\alpha^n \mapsto n$. Thus if $x_1, \ldots, x_k \in \mathbb{F}_q^\times$ with $x_i = \alpha^{n_i}$, then the isomorphism takes an element of the form $x_1^{d_1} \ldots x_k^{d_k} \in \mathbb{F}_q^\times$ to the element $d_1 n_1 + \ldots + d_k n_k$. It follows that $\operatorname{coker}\left(\begin{pmatrix} B_E^t - I \\ C_E^t \end{pmatrix} : (\mathbb{F}_q^\times)^{E^0_{\mathrm{reg}}} \to (\mathbb{F}_q^\times)^{E^0}\right)$ (where the groups are written multiplicatively) is isomorphic to $\operatorname{coker}\left(\begin{pmatrix} B_E^t - I \\ C_E^t \end{pmatrix} : \mathbb{Z}^{E^0_{\mathrm{reg}}}_{q-1} \to \mathbb{Z}^{E^0}_{q-1}\right)$ (where the groups are written additively). □

3.4 Computations of K-theory for Certain Leavitt Path Algebras

In this section we compute the K-groups of a Leavitt path algebra under certain hypotheses on the K-groups of the underlying field. This allows us to calculate the K-groups of a Leavitt path algebra over any algebraically closed field.

Lemma 3.4.1. *If G is an abelian group and D is a divisible subgroup of G, then $G \cong D \oplus G/D$.*

Proof. Since D is divisible, it follows that D is injective by [15, Ch.XX §4 Lemma 4.2]. Hence the identity map $i : D \to D$ extends to a homomorphism $r : G \to D$. Since $i : D \to D \subseteq G$ is a section for the short exact sequence $0 \to \ker r \to G \to D \to 0$, this sequence splits and $G \cong D \oplus \ker r$. It therefore suffices to prove that $\ker r$ is isomorphic to G/D.

To this end, define $h : \ker r \to G/D$ by $h(g) := g + D$. It is easily shown that h is a homomorphism. In addition, if $g \in \ker r$ with $h(g) = 0$, then $g \in D$ and $g = r(g) = 0$. Hence h is injective. If $g + D \in G/D$, then $g - r(g) \in \ker r$ and $h(g - r(g)) = (g - r(g)) + D = g + D$, so h is surjective. Hence h is an isomorphism and $\ker r \cong G/D$. □

Theorem 3.4.2. *Let E be a graph, let k be a field, and consider the Leavitt path algebra $L_{\mathsf{k}}(E)$. Given $n \in \mathbb{N}$, if $K_n(\mathsf{k})$ is divisible or if $K_{n-1}(\mathsf{k})$ is free abelian, then*

$K_n(L_{\mathsf{k}}(E))$ is isomorphic to a direct sum:

$$K_n(L_{\mathsf{k}}(E)) \cong \operatorname{coker}\left(\begin{pmatrix} B_E^t - I \\ C_E^t \end{pmatrix} : K_n(\mathsf{k})^{E^0_{\mathrm{reg}}} \to K_n(\mathsf{k})^{E^0}\right) \oplus \ker\left(\begin{pmatrix} B_E^t - I \\ C_E^t \end{pmatrix} : K_{n-1}(\mathsf{k})^{E^0_{\mathrm{reg}}} \to K_{n-1}(\mathsf{k})^{E^0}\right).$$

In particular, these hypotheses are satisfied and the isomorphism holds for all $n \in \mathbb{Z}$ when k is an algebraically closed field.

Proof. The long exact sequence of Theorem 3.1.1 induces the short exact sequence

$$0 \longrightarrow \operatorname{coker}\left(\begin{pmatrix} B_E^t - I \\ C_E^t \end{pmatrix} : K_n(\mathsf{k})^{E^0_{\mathrm{reg}}} \to K_n(\mathsf{k})^{E^0}\right) \longrightarrow K_n(L_{\mathsf{k}}(E)) \longrightarrow \ker\left(\begin{pmatrix} B_E^t - I \\ C_E^t \end{pmatrix} : K_{n-1}(\mathsf{k})^{E^0_{\mathrm{reg}}} \to K_{n-1}(\mathsf{k})^{E^0}\right) \longrightarrow 0.$$

Suppose $K_n(\mathsf{k})$ is divisible. Since direct sums of divisible groups and quotients of divisible groups are divisible, the cokernel is divisible and Lemma 3.4.1 implies that the conclusion of the theorem holds. On the other hand, if $K_{n-1}(\mathsf{k})$ is free abelian, then $\ker\begin{pmatrix} B_E^t - I \\ C_E^t \end{pmatrix}$ is free abelian. This implies the short exact sequence splits, and the conclusion of the theorem holds.

If k is an algebraically closed field, then $K_n(\mathsf{k})$ is divisible for each $n \geq 1$. This follows from [27, Theorem VI.1.6] if $\operatorname{char}(\mathsf{k}) = 0$, and from [27, Corollary VI.1.3.1] if $\operatorname{char}(\mathsf{k}) \neq 0$. Moreover, if $n \leq 0$, then $K_{n-1}((\mathsf{k}) = \{0\}$ is free abelian. Thus when k is an algebraically closed field, the hypotheses are satisfied and the isomorphism holds for all $n \in \mathbb{Z}^+$. □

Example 3.4.3. If E is the graph consisting of one vertex and n edges

and $n \geq 2$, then $L_{\mathsf{k}}(E)$ is isomorphic to the Leavitt algebra L_n. If $n = 1$, then $L_{\mathsf{k}}(E)$ is isomorphic to the Laurent polynomials $\mathsf{k}[x, x^{-1}]$. We consider $K_j(L_{\mathsf{k}}(E))$ when (i) k is algebraically closed with characteristic 0, (ii) $\mathsf{k} = \mathbb{R}$, and (iii) k has characteristic $p > 0$ and $n = p^m + 1$.

First suppose k is an algebraically closed field with characteristic 0. By [27, Theorem VI.1.6], $K_{2j}(\mathsf{k})$ is isomorphic to a uniquely divisible group, and $K_{2j-1}(\mathsf{k})$ is isomorphic to the direct sum of a uniquely divisible group and $\mathbb{Q}/\mathbb{Z}$, for $j \geq 1$. Here $\begin{pmatrix} B_E^t - I \\ C_E^t \end{pmatrix} = (n-1)$. Multiplication by a nonzero integer is an isomorphism on a uniquely divisible group, so if $n \neq 1$, Theorem 3.4.2 gives

$$\begin{aligned} &K_{2j}(L_{\mathsf{k}}(E)) \\ &\quad \cong \operatorname{coker}\left((n-1) : K_{2j}(\mathsf{k}) \to K_{2j}(\mathsf{k})\right) \oplus \ \ker\left((n-1) : K_{2j-1}(\mathsf{k}) \to K_{2j-1}(\mathsf{k})\right) \\ &\quad \cong \{0\} \oplus \ker((n-1) : \mathbb{Q}/\mathbb{Z} \to \mathbb{Q}/\mathbb{Z}) \cong \mathbb{Z}/(n-1)\mathbb{Z} \cong \mathbb{Z}_{n-1} \end{aligned}$$

(where for the last isomorphism one checks that the class in $\mathbb{Q}/\mathbb{Z}$ represented by $\frac{1}{n-1}$ generates $\ker((n-1) : \mathbb{Q}/\mathbb{Z} \to \mathbb{Q}/\mathbb{Z})$ with order $n-1$), and

$$\begin{aligned} &K_{2j+1}(L_{\mathsf{k}}(E)) \\ &\quad \cong \operatorname{coker}\left((n-1) : K_{2j+1}(\mathsf{k}) \to K_{2j+1}(\mathsf{k})\right) \oplus \ \ker\left((n-1) : K_{2j}(\mathsf{k}) \to K_{2j}(\mathsf{k})\right) \\ &\quad \cong \operatorname{coker}((n-1) : \mathbb{Q}/\mathbb{Z} \to \mathbb{Q}/\mathbb{Z}) \oplus \{0\} \cong \{0\} \end{aligned}$$

If $n = 1$, then $K_j(L_{\mathsf{k}}(E)) \cong K_j(\mathsf{k}) \oplus K_{j-1}(\mathsf{k})$ for all $j \in \mathbb{Z}$.

Now suppose $\mathsf{k} = \mathbb{R}$. By [27, Theorem VI.3.1] or [13, Corollary 22.6], if $j \in \mathbb{N}$ then

$$K_j(\mathbb{R}) \cong \begin{cases} D_j \oplus \mathbb{Z}_2 & \text{if } j \equiv 1, 2 \pmod 8 \\ D_j \oplus \mathbb{Q}/\mathbb{Z} & \text{if } j \equiv 3, 7 \pmod 8 \\ D_j & \text{if } j \equiv 0, 4, 5, 6 \pmod 8, \end{cases}$$

where D_j is a uniquely divisible group. If n is even, then

$$K_j(L_{\mathbb{R}}(E)) \cong \begin{cases} 0 & \text{if } j \equiv 1, 2, 3 \pmod 4 \\ \mathbb{Z}_{n-1} & \text{if } j \equiv 0 \pmod 4, \end{cases}$$

and if $n \neq 1$ is odd, then

$$K_j(L_{\mathbb{R}}(E)) \cong \begin{cases} 0 & \text{if } j \equiv 5, 6, 7 \pmod 8 \\ \mathbb{Z}_2 & \text{if } j \equiv 1, 3 \pmod 8 \\ \mathbb{Z}_{n-1} & \text{if } j \equiv 0, 4 \pmod 8 \\ \mathbb{Z}_4 \text{ or } \mathbb{Z}_2 \oplus \mathbb{Z}_2 & \text{if } j \equiv 2 \pmod 8 \end{cases}$$

for $j \in \mathbb{N}$.

Finally, suppose k is a field with characteristic $p > 0$. By [27, Theorem VI.4.7 (b)], it follows $K_j(\mathsf{k})$ has no p-torsion for $j \geq 0$. If $n = p^m + 1$ for some $m \in \mathbb{N}$, then using Theorem 3.1.1 and the fact that $\ker((p^m) : K_{j-1}(\mathsf{k}) \to K_{j-1}(\mathsf{k})) = 0$, we obtain

$$K_j(L_{\mathsf{k}}(E)) = \operatorname{coker}((p^m) : K_j(\mathsf{k}) \to K_j(\mathsf{k})) \cong K_j(\mathsf{k})/p^m K_j(\mathsf{k}).$$

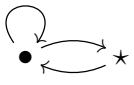
Definition 3.4.4 (The Cuntz Splice at a vertex v). Let $E = (E^0, E^1, r_E, s_E)$ be a graph and let $v \in E^0$. Define a graph $F = (F^0, F^1, r_F, s_F)$ by $F^0 = E^0 \cup \{v_1, v_2\}$, $F^1 = E^1 \cup \{e_1, e_2, f_1, f_2, h_1, h_2\}$, and let r_F and s_F extend r_E and s_E, respectively, and satisfy

$$s_F(e_1) = v, s_F(e_2) = v_1, s_F(f_1) = v_1, s_F(f_2) = v_2, s_F(h_1) = v_1, s_F(h_2) = v_2$$

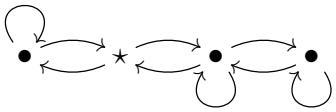
and

$$r_F(e_1) = v_1, r_F(e_2) = v, r_F(f_1) = v_2, r_F(f_2) = v_1, r_F(h_1) = v_1, r_F(h_2) = v_2.$$

We say that F is obtained by applying the Cuntz splice to E at v. For example, the graph



becomes



if we apply the Cuntz splice at the $\star$ vertex.

It was shown in [22, Proposition 9.3] that if E is a graph, then the Cuntz splice preserves the K_0-group and the K_1-group of the associated Leavitt path algebra $L_{\mathsf{k}}(E)$ for any choice of k. Here we show that, for the kinds of fields described in Theorem 3.4.2, the Cuntz splice preserves the K_n-group of the associated Leavitt path algebra for all $n \in \mathbb{Z}$.

Corollary 3.4.5. *Let E be a graph, let $v \in E^0$, and let F be a graph obtained by applying the Cuntz splice to E at v. If k is a field and $n \in \mathbb{Z}$ such that either $K_n(\mathsf{k})$ is divisible or $K_{n-1}(\mathsf{k})$ is free, then $K_n(L_{\mathsf{k}}(E)) \cong K_n(L_{\mathsf{k}}(F))$.*

Proof. We use an argument similar to the one in [22, Proposition 9.3(2)]. We begin by decomposing $E^0 = E^0_{\text{reg}} \sqcup E^0_{\text{sing}}$ and writing the vertex matrix of E as

$$A_E = \begin{pmatrix} B_E & C_E \\ * & * \end{pmatrix}.$$

Suppose v is a regular vertex. Then the vertex matrix of F has the form

$$A_F = \left(\begin{array}{cc|ccc|ccc} 1 & 1 & 0 & 0 & \cdots & 0 & 0 & \cdots \\ 1 & 1 & 1 & 0 & \cdots & 0 & 0 & \cdots \\ \hline 0 & 1 & & & & & & \\ 0 & 0 & & B_E & & & C_E & \\ \vdots & \vdots & & & & & & \\ \hline 0 & 0 & & & & & & \\ 0 & 0 & & * & & & * & \\ \vdots & \vdots & & & & & & \end{array}\right).$$

By Theorem 3.4.2, we obtain $K_n(L_{\mathsf{k}}(F))$ by considering the cokernel and kernel of the matrix

$$\left(\begin{array}{cc|ccc} 0 & 1 & 0 & 0 & \cdots \\ 1 & 0 & 1 & 0 & \cdots \\ \hline 0 & 1 & & & \\ 0 & 0 & & B_E^t - I & \\ \vdots & \vdots & & & \\ \hline 0 & 0 & & & \\ 0 & 0 & & C_E^t & \\ \vdots & \vdots & & & \end{array}\right) \quad \begin{array}{c}\text{equivalent} \\ \longleftrightarrow \end{array} \quad \left(\begin{array}{cc|ccc} 1 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & \cdots \\ \hline 0 & 0 & & & \\ 0 & 0 & & B_E^t - I & \\ \vdots & \vdots & & & \\ \hline 0 & 0 & & & \\ 0 & 0 & & C_E^t & \\ \vdots & \vdots & & & \end{array}\right).$$

The 2×2 identity in the upper-left-hand corner has no effect on the cokernel and kernel, so

$$K_n(L_{\mathsf{k}}(F)) \cong \operatorname{coker}\left(\begin{pmatrix} B_E^t - I \\ C_E^t \end{pmatrix} : K_n(\mathsf{k})^{E^0_{\mathrm{reg}}} \to K_n(\mathsf{k})^{E^0}\right)$$

$$\oplus \ \ker\left(\begin{pmatrix} B_E^t - I \\ C_E^t \end{pmatrix} : K_{n-1}(\mathsf{k})^{E^0_{\mathrm{reg}}} \to K_{n-1}(\mathsf{k})^{E^0}\right)$$

$$\cong K_n(L_{\mathsf{k}}(E)).$$

Next, suppose v is a singular vertex. Then the vertex matrix of F has the form

$$A_F = \left(\begin{array}{cc|ccc|ccc} 1 & 1 & 0 & 0 & \cdots & 0 & 0 & \cdots \\ 1 & 1 & 0 & 0 & \cdots & 1 & 0 & \cdots \\ \hline 0 & 0 & & & & & & \\ 0 & 0 & & B_E & & & C_E & \\ \vdots & \vdots & & & & & & \\ \hline 0 & 1 & & & & & & \\ 0 & 0 & & * & & & * & \\ \vdots & \vdots & & & & & & \end{array}\right).$$

By Theorem 3.4.2, we obtain $K_n(L_{\mathsf{k}}(F))$ by considering the cokernel and kernel of the matrix

$$\left(\begin{array}{cc|ccc} 0 & 1 & 0 & 0 & \cdots \\ 1 & 0 & 0 & 0 & \cdots \\ \hline 0 & 0 & & & \\ 0 & 0 & & B_E^t - I & \\ \vdots & \vdots & & & \\ \hline 0 & 1 & & & \\ 0 & 0 & & C_E^t & \\ \vdots & \vdots & & & \end{array}\right) \quad \begin{array}{c} \text{equivalent} \\ \longleftrightarrow \end{array} \quad \left(\begin{array}{cc|ccc} 1 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & \cdots \\ \hline 0 & 0 & & & \\ 0 & 0 & & B_E^t - I & \\ \vdots & \vdots & & & \\ \hline 0 & 0 & & & \\ 0 & 0 & & C_E^t & \\ \vdots & \vdots & & & \end{array}\right).$$

The 2×2 identity in the upper-left-hand corner has no effect on the cokernel and kernel, so as above $K_n(L_{\mathsf{k}}(F)) \cong K_n(L_{\mathsf{k}}(E))$. □

The following theorem is inspired by [6, Theorem 9.4].

Theorem 3.4.6. *Let E be a finite graph with no sinks such that $\det(A_E^t - I) \neq 0$. If k is an algebraically closed field, then*

$$K_n(L_{\mathsf{k}}(E)) \cong \begin{cases} 0 & \text{if } n \geq 0 \text{ is odd} \\ \ker((A_E^t - I) : G^{E^0} \to G^{E^0}) & \text{if } n \geq 0 \text{ is even,} \end{cases}$$

where $G := \mathbb{Q}/\mathbb{Z}$ if $\mathrm{char}(\mathsf{k}) = 0$ or $G := \mathbb{Q}/\mathbb{Z}[\frac{1}{p}]$ if $\mathrm{char}(\mathsf{k}) = p > 0$.

Note that this produces a weak "Bott periodicity" for Leavitt path algebras over algebraically closed fields and with $\det(A_E^t - I) \neq 0$: Under these hypotheses we have that $K_{2n}(L_{\mathsf{k}}(E)) \cong K_2(L_{\mathsf{k}}(E))$ and $K_{2n-1}(L_{\mathsf{k}}(E)) \cong K_1(L_{\mathsf{k}}(E)) \cong 0$ for all $n \in \mathbb{N}$. Moreover, if $\mathrm{char}(\mathsf{k}) \nmid \det(A_E^t - I)$, then we also have $K_0(L_{\mathsf{k}}(E)) \cong K_2(L_{\mathsf{k}}(E))$, so that $K_{2n}(L_{\mathsf{k}}(E)) \cong K_0(L_{\mathsf{k}}(E))$ for all $n \in \mathbb{Z}^+$

Proof. We may write

$$K_n(\mathsf{k}) \cong \begin{cases} D_n \oplus G & \text{if } n \geq 0 \text{ is odd} \\ D_n & \text{if } n \geq 0 \text{ is even,} \end{cases}$$

where D_n is a uniquely divisible group, $G = \mathbb{Q}/\mathbb{Z}$ if $\mathrm{char}(\mathsf{k}) = 0$ by [27, Theorem VI.1.6], and $G = \mathbb{Q}/\mathbb{Z}[\frac{1}{p}]$ if $\mathrm{char}(\mathsf{k}) = p > 0$ by [27, Corollary VI.1.3.1]. In either case G is divisible.

Since $(A_E^t - I)$ is an $E^0 \times E^0$ matrix with nonzero determinant, it has Smith normal form with nonzero diagonal entries and zeros elsewhere. Since D_n is uniquely

divisible, $(A_E^t - I) : D_n^{E^0} \to D_n^{E^0}$ is an isomorphism, and since G is divisible, $(A_E^t - I) : G^{E^0} \to G^{E^0}$ is a surjection. If n is odd, the map

$$(A_E^t - I) : K_n(\mathsf{k})^{E^0} \to K_n(\mathsf{k})^{E^0}$$

decomposes as

$$(A_E^t - I) \oplus (A_E^t - I) : D_n^{E^0} \oplus G^{E^0} \to D_n^{E^0} \oplus G^{E^0},$$

and is an isomorphism in the first summand and a surjection in the second. Thus, when we apply Theorem 3.4.2, if n is odd we obtain $K_n(L_{\mathsf{k}}(E)) \cong \{0\} \oplus \{0\}$, and if n is even we obtain $K_n(L_{\mathsf{k}}(E)) \cong \{0\} \oplus \ker\left((A_E^t - I) : G^{E^0} \to G^{E^0}\right)$.

In addition, for $n = 0$ we have $K_0(L_{\mathsf{k}}(E)) \cong \operatorname{coker}((A_E^t - I) : \mathbb{Z}^{E^0} \to \mathbb{Z}^{E^0})$ by Proposition 3.2.2.

Finally, suppose that the additional hypothesis $\operatorname{char}(\mathsf{k}) \nmid \det(A_E^t - I)$ holds. Let $\operatorname{diag}(n_1, \ldots, n_s)$ be the Smith normal form for the matrix $A_E^t - I$. Then $\det(A_E^t - I) = n_1 \ldots n_s$, and since $\operatorname{char}(\mathsf{k}) \nmid \det(A_E^t - I)$, we may conclude that $\operatorname{char}(\mathsf{k}) \nmid n_i$ for all $1 \leq i \leq s$. If we consider the multiplication $(n_i) : G \to G$, a straightforward computation shows that this map has kernel equal to $\left\langle \overline{\frac{1}{n_i}} \right\rangle$, where $\overline{\frac{1}{n_i}}$ is the class in G represented by the element $\frac{1}{n_i}$ and $\left\langle \overline{\frac{1}{n_i}} \right\rangle$ is the additive subgroup generated by $\overline{\frac{1}{n_i}}$. Since this is a cyclic group of order n_i, we have that $(\ker(n_i) : G \to G) \cong \mathbb{Z}_{n_i}$, and therefore,

$$\ker((A_E^t - I) : G^{E^0} \to G^{E^0}) \cong \mathbb{Z}_{n_1} \oplus \cdots \oplus \mathbb{Z}_{n_s} \cong \operatorname{coker}((A_E^t - I) : \mathbb{Z}^{E^0} \to \mathbb{Z}^{E^0}).$$

□

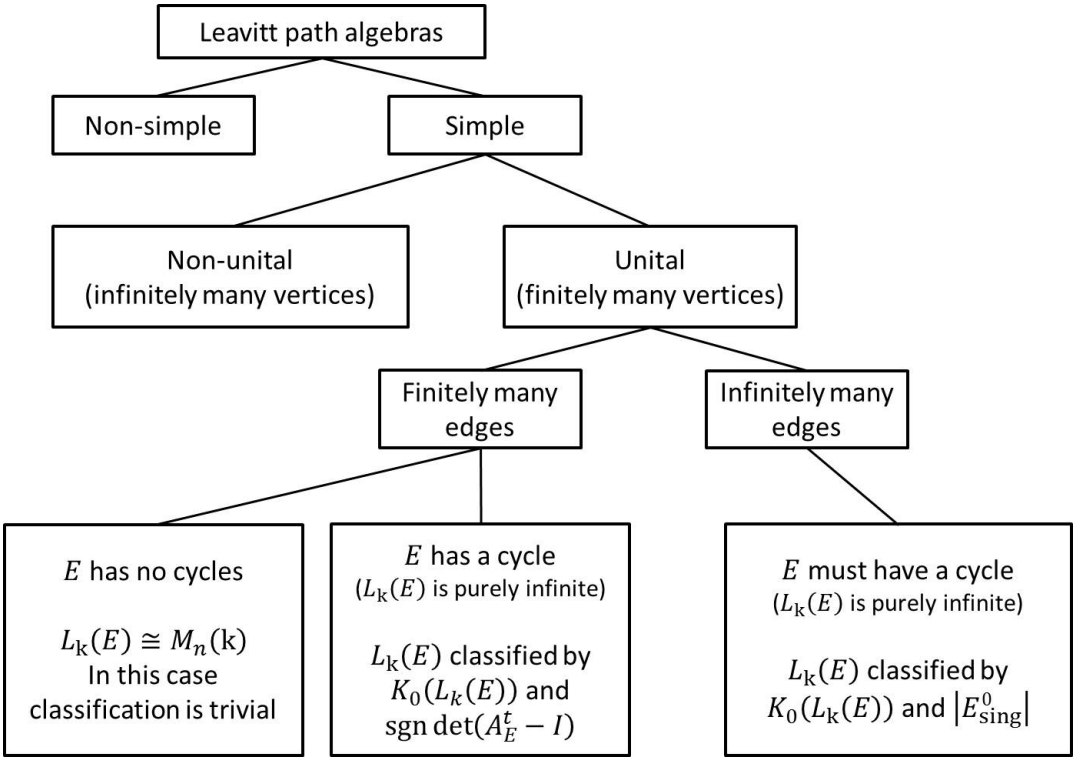
CHAPTER 4

Rank, Corank, and Classification

We have seen that we can use the long exact sequence to compute the algebraic K-groups of a Leavitt path algebra. We also wish to know when the algebraic K-groups provide a complete Morita equivalence invariant for classification, which is the subject of this chapter.

We assume our Leavitt path algebras are *simple*. It is a standard assumption in classification, because we think of the simple objects as the "building blocks" of all the objects. Next, we choose to consider *unital* Leavitt path algebras. This is equivalent to requiring the graph to have only finitely many vertices. In particular, in this case the adjacency matrix is finite-dimensional. Then, we decide whether

there are only finitely many edges or infinitely many edges, which turns out to be a major dividing line. These choices are summarized in the following chart:



The bottom-left box summarizes the discussion up to and including [22, Lemma 8.1]. The bottom-center box is [3, Theorem 1.25]. The bottom-right box is [22, Theorem 8.6 (3)]:

Theorem 4.0.7. *Let* k *be any field, and let* $L_{\mathsf{k}}(E)$ *and* $L_{\mathsf{k}}(F)$ *be unital simple Leavitt path algebras. If* E *and* F *both have an infinite number of edges, then* $L_{\mathsf{k}}(E)$ *is Morita equivalent to* $L_{\mathsf{k}}(F)$ *if and only if* $K_0(L_{\mathsf{k}}(E)) \cong K_0(L_{\mathsf{k}}(F)$ *and* $|E^0_{\mathrm{sing}}| = |F^0_{\mathrm{sing}}|$.

On the one hand, this is a wonderful classification result. We know the formula for the K_0 -group in this case, and we can easily look at a graph and count the number of singular vertices. On the other hand, the number of singular vertices is not an intrinsic property of the algebra. Different graphs can yield isomorphic Leavitt path algebras, so any time we associate a graph with a Leavitt path algebra, it represents a choice we made. Somewhat analagous is a choice of basis: it's "artificial." Moreover, the "number of singular vertices" does not make sense for general algebras. We wish to replace this condition with an intrinsically algebraic property (such as algebraic K-groups).

In this chapter we develop a strategy for finding when the "number of singular vertices" condition can be replaced with an algebraic condition, and the main considerations depend on the underlying field. At first this is remarkable, because the field did not play a role in any of the previous cases. However, the long exact sequence (discussed in the previous chapter) relates the K_n-group of the Leavitt path algebra to the K_n-group and K_{n-1}-group of the underlying field, so it is less surprising that eventually the field comes in to play.

4.1 Rank and Corank of an Abelian Group

In this section we examine and develop basic properties of the rank and corank of an abelian group. We also compare the values that the rank and corank can assign to an abelian group.

4.1.1 Rank of an Abelian Group

Definition 4.1.1. If $(G, +)$ is an abelian group, a finite collection of elements $\{g_i\}_{i=1}^k \subseteq G$ is *linearly independent (over $\mathbb{Z}$)* if whenever $\sum_{i=1}^k n_i g_i = 0$ for $n_1, \ldots, n_k \in \mathbb{Z}$, then $n_1 = \ldots = n_k = 0$. Any two maximal linearly independent sets in G have the same cardinality, and we define $\operatorname{rank} G$ to be this cardinality if a maximal linearly independent set exists and ∞ otherwise.

Remark 4.1.2. Let G be an abelian group. One can see that if G contains a linearly independent set with n elements, then there exists an injective homomorphism $\iota : \mathbb{Z}^n \to G$ (given a linearly independent set of n elements in G, the fact $\mathbb{Z}^n$ is free abelian gives a homomorphism taking the generators of $\mathbb{Z}^n$ to these elements and the linear independence implies this homomorphism is injective). Conversely, any injective homomorphism $\iota : \mathbb{Z}^n \to G$ will send the generators of $\mathbb{Z}^n$ to a set of n linearly independent elements in G. Thus

$$\operatorname{rank} G = \sup\{n \in \mathbb{Z}^+ : \text{there exists an injective homomorphism } \iota : \mathbb{Z}^n \to G\}. \tag{4.1}$$

Furthermore, if we form the tensor product $\mathbb{Q} \otimes_{\mathbb{Z}} G$, then since $\mathbb{Q}$ is a field, $\mathbb{Q} \otimes_{\mathbb{Z}} G$ is a vector space, and maximal linearly independent sets in G correspond to bases in $\mathbb{Q} \otimes_{\mathbb{Z}} G$. Thus

$$\operatorname{rank} G = \dim_{\mathbb{Q}}(\mathbb{Q} \otimes_{\mathbb{Z}} G) \tag{4.2}$$

where $\dim_{\mathbb{Q}}$ denotes the dimension as a $\mathbb{Q}$-vector space.

The equations in (4.1) and (4.2) give two equivalent ways to define the rank of an abelian group.

Remark 4.1.3. It is important to notice that we are working in the category of abelian groups and defining the "rank of an abelian group". This is different from how the "rank of a group" is defined: If G is a (not necessarily abelian) group, then the rank of G is defined to be the smallest cardinality of a generating set for G. These notions do not coincide; for example the group-rank of $\mathbb{Z}_n$ is 1, while using the abelian-group-rank from Definition 4.1.1 we have $\operatorname{rank}\mathbb{Z}_n = 0$. Sometime the term "torsion-free rank" or "Prüfer rank" is used for this abelian-group-rank; however, we are going to simply call it "rank" with the understanding we are working in the category of abelian groups.

The following are some well-known facts about the torsion-free rank of an abelian group. We will use these facts in the next sections.

Proposition 4.1.4. *The rank of an abelian group satisfies the following elementary properties:*

(i) *If G and H are isomorphic abelian groups, then* $\operatorname{rank} G = \operatorname{rank} H$.

(ii) $\operatorname{rank}\mathbb{Z} = 1$

(iii) $\operatorname{rank} G = 0$ *if and only if G is a torsion group*

(iv) *If $0 \to P \to Q \to R \to 0$ is an exact sequence of abelian groups P, Q, and R, then* $\operatorname{rank} Q = \operatorname{rank} P + \operatorname{rank} R$.

(v) *If G_1 and G_2 are abelian groups, then* $\operatorname{rank}(G_1 \oplus G_2) = \operatorname{rank} G_1 + \operatorname{rank} G_2$

(vi) *If* $\operatorname{rank} G = n < \infty$, *then there exists an injective homomorphism $\iota : \mathbb{Z}^n \to G$, and if* $\operatorname{rank} G = \infty$ *there exists an injective homomorphism $\iota : \bigoplus_{i=1}^{\infty} \mathbb{Z} \to G$.*

Proof. Items (i)–(v) follow from well-known properties of vector spaces over a field and the fact that $\mathbb{Q}$ is a flat $\mathbb{Z}$-module. For (vi), if $\operatorname{rank} G < \infty$ the result follows from (4.1). If $\operatorname{rank} G = \infty$, then (4.2) implies $\mathbb{Q} \otimes_{\mathbb{Z}} G$ is an infinite-dimensional vector space over $\mathbb{Q}$ and hence contains an infinite basis, which corresponds to a infinite set in G for which every finite subset is linearly independent. The fact $\bigoplus_{i=1}^{\infty} \mathbb{Z}$ is a free abelian group with countably many generators implies there exists a homomorphism $\iota : \bigoplus_{i=1}^{\infty} \mathbb{Z} \to G$ and the fact every finite subset is linearly independent implies ι is injective.

□

4.1.2 Corank of an Abelian Group

In analogy with the equation for the rank of an abelian group derived in (4.1), we make the following definition.

Definition 4.1.5. If G is an abelian group we define the *corank of* G to be

$$\operatorname{corank} G := \sup\{n \in \mathbb{Z}^+ : \text{there exists a surjective homomorphism } \pi : G \to \mathbb{Z}^n\}.$$

Note that $\operatorname{corank} G$ is an element of the extended positive integers $\{0, 1, 2, \ldots, \infty\}$.

Remark 4.1.6. If G is an abelian group and $\pi : G \to \mathbb{Z}^n$ is a surjective homomorphism, then $G/\ker \pi \cong \mathbb{Z}^n$. Conversely, if N is a subgroup of G with $G/N \cong \mathbb{Z}^n$, then the quotient map from G onto G/N composed with an isomorphism from G/N onto $\mathbb{Z}^n$ is surjective. Hence

$$\operatorname{corank} G = \sup\{\operatorname{rank}(G/N) : N \text{ is a subgroup of } G \text{ and } G/N \text{ is free abelian}\}. \tag{4.3}$$

Remark 4.1.7. If G is an abelian group and $\operatorname{corank} G = \infty$, then Definition 4.1.5 implies that for every $n \in \mathbb{Z}$ there exists a surjective homomorphism from G onto $\mathbb{Z}^n$. However, if $\operatorname{corank} G = \infty$ it is not necessarily true that there exists a surjective homomorphism from G onto $\bigoplus_{i=1}^{\infty} \mathbb{Z}$. For example, if we consider the infinite direct product $\prod_{i=1}^{\infty} \mathbb{Z}$, then we see that projecting onto the first n coordinates gives a surjective homomorphism onto $\mathbb{Z}^n$, and hence $\operatorname{corank} \prod_{i=1}^{\infty} \mathbb{Z} = \infty$. However, there is no surjective homomorphism from $\prod_{i=1}^{\infty} \mathbb{Z}$ onto $\bigoplus_{i=1}^{\infty} \mathbb{Z}$ (The reason for this is that $\bigoplus_{i=1}^{\infty} \mathbb{Z}$ is a "slender" group, see [11, Chapter VIII, §94].) Contrast this situation with what occurs when $\operatorname{rank} G = \infty$ in Proposition 4.1.4(vi).

Definition 4.1.8. Recall that an abelian group G is *divisible* if for every $y \in G$ and for every $n \in \mathbb{N}$, there exists $x \in G$ such that $nx = y$. Likewise, an abelian group G is *weakly divisible* if for every $y \in G$ and for every $N \in \mathbb{N}$, there exists $n \geq N$ and $x \in G$ such that $nx = y$.

Proposition 4.1.9. *The corank of an abelian group satisfies the following elementary properties:*

(i) If G and H are isomorphic abelian groups, then $\operatorname{corank} G = \operatorname{corank} H$.

(ii) $\operatorname{corank} \mathbb{Z}^n = n$.

(iii) If G is a torsion group, then $\operatorname{corank} G = 0$.

(iv) If G is a weakly divisible group, then $\operatorname{corank} G = 0$.

(v) $\operatorname{corank} G = 0$ *if and only if G has no nonzero free abelian quotients.*

Proof. The fact in (i) follows immediately from the definition of corank. For (ii) we see that the identity map is a surjective homomorphism from $\mathbb{Z}^n$ onto $\mathbb{Z}^n$ and observe that there are no surjective homomorphisms from $\mathbb{Z}^n$ onto $\mathbb{Z}^m$ when $m > n$. For (iii) observe that if G is a torsion group, then since homomorphisms take elements of finite order to elements of finite order, there are no nonzero homomorphisms from G to $\mathbb{Z}^n$, and hence $\operatorname{corank} G = 0$. For (iv) we observe that the homomorphic image of a weakly divisible group is weakly divisible, and since $\mathbb{Z}^n$ has no weakly divisible subgroups other than zero, $\operatorname{corank} G = 0$. For (v) we see from (4.3) that $\operatorname{corank} G = 0$ if and only if every free abelian quotient of G is the zero group. □

In addition to these facts, there are two important properties of corank that are not immediate that we establish in Proposition 4.1.10 and Proposition 4.1.13.

Proposition 4.1.10. *If G is an abelian group and H is a subgroup of G with* $\operatorname{corank} H = 0$, *then* $\operatorname{corank}(G/H) = \operatorname{corank} G$.

Proof. Since the quotient homomorphism $q : G \to G/N$ is surjective, we see that any surjective homomorphism $\pi : G/N \to \mathbb{Z}^n$ may be precomposed with q to obtain a surjective homomorphism $\pi \circ q : G \to \mathbb{Z}^n$. Hence $\operatorname{corank}(G/H) \leq \operatorname{corank} G$.

To obtain the inequality in the other direction, suppose that $n \in \mathbb{Z}^+$ and $\pi : G \to \mathbb{Z}^n$ is a surjective homomorphism. If $N := \ker \pi$, then $G/N \cong \mathbb{Z}^n$. Since $(H + N)/N$ is a subgroup of G/N, and subgroups of free abelian groups are free abelian, $(H + N)/N$ is free abelian. Since $(H + N)/N \cong H/(H \cap N)$, we have that $H/(H \cap N)$ is a free abelian group. Since $\operatorname{corank} H = 0$, it follows from (4.3) that H contains no nonzero free quotients. Hence $H/(H \cap N) = 0$, and $H \cap N = H$, so that

$H \subseteq N$. Since $(G/H)/(N/H) \cong G/N \cong \mathbb{Z}^n$, there is a surjective homomorphism from G/H onto $\mathbb{Z}^n$. It follows that $\operatorname{corank} G \leq \operatorname{corank}(G/H)$. □

Lemma 4.1.11. *If G_1 and G_2 are abelian groups and $\operatorname{corank} G_2 = 0$, then $\operatorname{corank}(G_1 \oplus G_2) = \operatorname{corank} G_1$.*

Proof. Since $\operatorname{corank}(0 \oplus G_2) = \operatorname{corank} G_2 = 0$, Proposition 4.1.10 implies that

$$\operatorname{corank}(G_1 \oplus G_2) = \operatorname{corank}((G_1 \oplus G_2)/(0 \oplus G_2)) = \operatorname{corank}(G_1 \oplus 0) = \operatorname{corank} G_1.$$

□

Lemma 4.1.12. *If $\operatorname{corank} G = n < \infty$, then $G \cong \mathbb{Z}^n \oplus H$ for an abelian group H with $\operatorname{corank} H = 0$.*

Proof. By the definition of corank there exists a surjective homomorphism $\pi : G \to \mathbb{Z}^n$. Let $H := \ker \pi$. Then $G/H \cong \mathbb{Z}^n$. Since G/H is a free abelian group, the short exact sequence $0 \to H \to G \to G/H \to 0$ splits and $G \cong G/H \oplus H \cong \mathbb{Z}^n \oplus H$.

Since $G \cong \mathbb{Z}^n \oplus H$, there is a surjective homomorphism from G onto H, and the fact that $\operatorname{corank} G < \infty$ implies $\operatorname{corank} H < \infty$. Let $m = \operatorname{corank} H < \infty$. Since m is finite there exists a surjective homomorphism $\pi' : H \to \mathbb{Z}^m$. As above, if we let $N := \ker \pi'$, then $H/N \cong \mathbb{Z}^m$, and since H/N is a free abelian group, the short exact sequence $0 \to N \to H \to H/N \to 0$ splits and $H \cong H/N \oplus N \cong \mathbb{Z}^m \oplus N$. Thus $G \cong \mathbb{Z}^n \oplus H \cong \mathbb{Z}^n \oplus \mathbb{Z}^m \oplus N$, and there is a surjective homomorphism from G onto $\mathbb{Z}^{n+m}$. Hence $n + m \leq \operatorname{corank} G = n$, and since m and n are non-negative, we conclude that $m = 0$. □

Proposition 4.1.13. *If G_1 and G_2 are abelian groups, then*

$$\operatorname{corank}(G_1 \oplus G_2) = \operatorname{corank} G_1 + \operatorname{corank} G_2.$$

Proof. If $\operatorname{corank} G_1 = \infty$, then for every $n \in \mathbb{N}$ there exists a surjective homomorphism $\pi : G_1 \to \mathbb{Z}^n$. If we precompose with the projection onto the first coordinate, $\pi_1 : G_1 \oplus G_2 \to G_1$ given by $\pi_1(g_1, g_2) := g_1$, then $\pi \circ \pi_1 : G_1 \oplus G_2 \to \mathbb{Z}^n$ is surjective. It follows that $\operatorname{corank}(G_1 \oplus G_2) = \infty$. Thus $\operatorname{corank}(G_1 \oplus G_2) = \infty = \infty + \operatorname{corank} G_2 = \operatorname{corank} G_1 + \operatorname{corank} G_2$.

A similar argument shows that if $\operatorname{corank} G_2 = \infty$, then $\operatorname{corank}(G_1 \oplus G_2) = \infty$ and $\operatorname{corank}(G_1 \oplus G_2) = \operatorname{corank} G_1 + \operatorname{corank} G_2$.

If $\operatorname{corank} G_1 = n_1 < \infty$ and $\operatorname{corank} G_2 = n_2 < \infty$, then Lemma 4.1.12 implies that $G_1 \cong \mathbb{Z}^{n_1} \oplus H_1$ and $G_2 \cong \mathbb{Z}^{n_2} \oplus H_2$ for some abelian groups H_1 and H_2 with $\operatorname{corank} H_1 = \operatorname{corank} H_2 = 0$. Thus

$$\begin{aligned}
\operatorname{corank}(G_1 \oplus G_2) &= \operatorname{corank}((\mathbb{Z}^{n_1} \oplus H_1) \oplus (\mathbb{Z}^{n_2} \oplus H_2)) \\
&= \operatorname{corank}(\mathbb{Z}^{n_1} \oplus \mathbb{Z}^{n_2} \oplus H_1 \oplus H_2) \\
&= \operatorname{corank}(\mathbb{Z}^{n_1} \oplus \mathbb{Z}^{n_2} \oplus H_1) \qquad \text{(by Lemma 4.1.11)} \\
&= \operatorname{corank}(\mathbb{Z}^{n_1} \oplus \mathbb{Z}^{n_2}) \qquad \text{(by Lemma 4.1.11)} \\
&= n_1 + n_2 \\
&= \operatorname{corank} G_1 + \operatorname{corank} G_2.
\end{aligned}$$

□

4.1.3 A Comparison of Rank and Corank

We begin by computing the rank and corank of some groups to observe that their values do not always agree.

Example 4.1.14. If $\mathbb{Q}$ denotes the abelian group of rational numbers with addition, then we see that $\operatorname{corank}\mathbb{Q} = 0$ since $\mathbb{Q}$ is divisible. In addition, $\operatorname{rank}\mathbb{Q} \geq 1$ since $\mathbb{Q}$ is a not a torsion group, and for any set of two elements $\{m, n\} \subseteq \mathbb{Q}$, we have $n(m) - m(n) = 0$ so that $\{m, n\}$ is linearly dependent. Hence $\operatorname{rank}\mathbb{Q} = 1$.

If $\mathbb{R}$ denotes the abelian group of real numbers with addition, then we see that $\operatorname{corank}\mathbb{R} = 0$ since $\mathbb{R}$ is divisible. For any prime p, the set of square roots of prime numbers up to p, namely $\{\sqrt{2}, \sqrt{3}, \sqrt{5}, \sqrt{7}, \sqrt{11}, \ldots, \sqrt{p}\}$, is a linearly independent subset of $\mathbb{R}$, so $\operatorname{rank}\mathbb{R} = \infty$.

If we let $\prod_{i=1}^{\infty}\mathbb{Z}$ be the product of countably many copies of $\mathbb{Z}$, then for any $n \in \mathbb{N}$ there is an injection $\iota_n : \mathbb{Z}^n \to \prod_{i=1}^{\infty}\mathbb{Z}$ obtained by including into the first n coordinates, and there is surjection $\pi_n : \prod_{i=1}^{\infty}\mathbb{Z} \to \mathbb{Z}^n$ obtained by projecting onto the first n coordinates. Hence $\operatorname{rank}\prod_{i=1}^{\infty}\mathbb{Z} = \operatorname{corank}\prod_{i=1}^{\infty}\mathbb{Z} = \infty$.

We display these results here for easy reference:

$$\operatorname{corank}\mathbb{Q} = 0 \qquad \operatorname{rank}\mathbb{Q} = 1$$

$$\operatorname{corank}\mathbb{R} = 0 \qquad \operatorname{rank}\mathbb{R} = \infty$$

$$\operatorname{corank}\prod_{i=1}^{\infty}\mathbb{Z} = \infty \qquad \operatorname{rank}\prod_{i=1}^{\infty}\mathbb{Z} = \infty$$

Example 4.1.15. Let $\iota : \mathbb{Z} \to \mathbb{Q}$ be the inclusion map. Note that $0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$ is a short exact sequence. However, $\operatorname{corank}\mathbb{Z} = 1$ and $\operatorname{corank}\mathbb{Q} = 0$, so

that $\operatorname{corank}\mathbb{Q} \neq \operatorname{corank}\mathbb{Z} + \operatorname{corank}(\mathbb{Q}/\mathbb{Z})$. Contrast this with the property of rank described in Proposition 4.1.4(iv).

Although the above examples show that rank and corank do not agree in general, the following proposition shows that they do agree on finitely generated abelian groups and on free abelian groups.

Proposition 4.1.16. *Let G be an abelian group such that $G \cong T \oplus F$, where T is a torsion group and F is a free group. Then* $\operatorname{rank} G = \operatorname{corank} G = \operatorname{rank} F$.

Proof. Proposition 4.1.4 implies that $\operatorname{rank} G = \operatorname{rank} T + \operatorname{rank} F = \operatorname{rank} F$. Proposition 4.1.13 and Proposition 4.1.9 imply that $\operatorname{corank} G = \operatorname{corank} T + \operatorname{corank} F = \operatorname{corank} F$. Since F is a free abelian group, $F \cong \bigoplus_{i \in I} \mathbb{Z}$. Thus $\operatorname{rank} F = |I|$ if I is finite, and $\operatorname{rank} F = \infty$ if I is infinite. Likewise, $\operatorname{corank} F = |I|$ if I is finite, and $\operatorname{corank} F = \infty$ if I is infinite. Hence, $\operatorname{rank} G = \operatorname{rank} F = \operatorname{corank} G$. □

The next proposition gives further insight into the relationship between rank and corank for general abelian groups. It also shows that the problem of finding an abelian group with unequal rank and corank is tantamount to finding an abelian group with corank zero and nonzero rank.

Proposition 4.1.17. *If G is an abelian group, then* $\operatorname{corank} G \leq \operatorname{rank} G$. *Furthermore, if* $\operatorname{rank} G \neq \operatorname{corank} G$, *then* $\operatorname{corank} G < \infty$ *and* $G \cong \mathbb{Z}^n \oplus H$, *where* $n = \operatorname{corank} G$ *and H is an abelian group with* $\operatorname{corank} H = 0$ *and* $\operatorname{rank} H = \operatorname{rank} G - n$.

Proof. For any $n \in \mathbb{Z}^+$, if there is a surjection $\pi : G \to \mathbb{Z}^n$, then $0 \to \ker \pi \to G \to \mathbb{Z}^n \to 0$ is a short exact sequence that splits (due to the fact $\mathbb{Z}^n$ is free) implying

that $G \cong \mathbb{Z}^n \oplus \ker \pi$. Hence, by Proposition 4.1.4, $\operatorname{rank} G = n + \operatorname{rank} \ker \pi \geq n$. This implies $\operatorname{corank} G \leq \operatorname{rank} G$.

If $\operatorname{rank} G \neq \operatorname{corank} G$, then the previous paragraph implies that $\operatorname{corank} G$ is finite. Hence by Lemma 4.1.12 we have $G \cong \mathbb{Z}^n \oplus H$ for an abelian group H with $\operatorname{corank} H = 0$. Thus $\operatorname{rank} G = n + \operatorname{rank} H$, and $\operatorname{rank} H = \operatorname{rank} G - n$. □

Proposition 4.1.17 shows that to find abelian groups with unequal rank and corank, one needs to focus on finding abelian groups with zero corank and positive rank.

Example 4.1.18. If $0 \leq m < n \leq \infty$, we may define $G := \mathbb{Z}^m \oplus \mathbb{Q}^{n-m}$. Then, using the computations from Example 4.1.14 we see that $\operatorname{corank} G = m + 0 = m$ and $\operatorname{rank} G = m + (n - m) = n$. Thus for any $0 \leq m < n \leq \infty$ there exists an abelian group G with $\operatorname{corank} G = m$ and $\operatorname{rank} G = n$.

Proposition 4.1.17 shows we must have the corank of an abelian group less than or equal to the rank, but this example shows that for all values $0 \leq m < n \leq \infty$ there exists an abelian group G with $\operatorname{corank} G = m$ and $\operatorname{rank} G = n$. Moreover, Proposition 4.1.17 implies that any such examples must be of the form $\mathbb{Z}^n \oplus H$ with $\operatorname{corank} H = 0$ and $\operatorname{rank} H \geq 1$.

4.2 Size Functions on Abelian Groups

Definition 4.2.1. A *size function* on the class of abelian groups is an assignment

$$F : \mathbf{Ab} \to \mathbb{Z}^+ \cup \{\infty\}$$

from the class of abelian groups **Ab** to the extended non-negative integers $\mathbb{Z}^+ \cup \{\infty\} = \{0, 1, 2, \ldots, \infty\}$ satisfying the following conditions:

(1) If G_1 and G_2 are abelian groups with $G_1 \cong G_2$, then $F(G_1) = F(G_2)$.

(2) If G is a torsion group, then $F(G) = 0$.

(3) If G is an abelian group and H is a subgroup of G with $F(H) = 0$, then $F(G/H) = F(G)$.

(4) If G_1 and G_2 are abelian groups, then $F(G_1 \oplus G_2) = F(G_1) + F(G_2)$.

Definition 4.2.2. An *exact size function* on the class of abelian groups is an assignment

$$F : \mathbf{Ab} \to \mathbb{Z}^+ \cup \{\infty\}$$

from the class of abelian groups **Ab** to the extended non-negative integers $\mathbb{Z}^+ \cup \{\infty\} = \{0, 1, 2, \ldots, \infty\}$ satisfying the following conditions:

(1) If G_1 and G_2 are abelian groups with $G_1 \cong G_2$, then $F(G_1) = F(G_2)$.

(2) If G is a torsion group, then $F(G) = 0$.

(3) If P, Q, and R are abelian groups and $0 \to P \to Q \to R \to 0$ is an exact sequence, then $F(Q) = F(P) + F(R)$.

Remark 4.2.3. We point out that in both Definition 4.2.1 and Definition 4.2.2 the domain of the assignment is a class (and not a set). Thus, despite the name, size functions and exact size functions are technically assignments and not functions.

Observe that *a priori* properties (3) and (4) of Definition 4.2.1 are not required to hold for an exact size function. The following proposition shows that, despite this, they do follow.

Proposition 4.2.4. *Any exact size function is also a size function.*

Proof. It suffices to show that any exact size function satisfies properties (3) and (4) of Definition 4.2.1. To establish (3), let G be an abelian group and let H be a subgroup of G with $F(H) = 0$. Then $0 \to H \to G \to G/H \to 0$ is exact, and hence $F(G) = F(H) + F(G/H) = 0 + F(G/H) = F(G/H)$. To establish (4), suppose G_1 and G_2 are abelian groups, and consider the exact sequence $0 \to G_1 \oplus 0 \to G_1 \oplus G_2 \to 0 \oplus G_2 \to 0$. Then $F(G_1 \oplus G_2) = F(G_1 \oplus 0) + F(0 \oplus G_2) = F(G_1) + F(G_2)$. □

Although any exact size function is a size function, the converse does not hold (see Example 4.2.9). However, any size function will satisfy the following special case of exactness.

Lemma 4.2.5. *Let $F : \mathbf{Ab} \to \mathbb{Z}^+ \cup \{\infty\}$ be a size function. If P, Q, and R are abelian groups, $0 \to P \to Q \to R \to 0$ is an exact sequence, and $F(P) = 0$, then $F(Q) = F(R)$.*

Proof. Due to the exactness of the sequence there is a subgroup H of Q such that $P \cong H$ and $Q/H \cong R$. Thus, using the properties of a size function, we see that $F(H) = F(P) = 0$, and hence $F(R) = F(Q/H) = F(Q)$. □

The following proposition shows that on finitely generated abelian groups a size function is a constant multiple of the rank function.

Proposition 4.2.6. *If $F : \mathbf{Ab} \to \mathbb{Z}^+ \cup \{\infty\}$ is a size function, and $k := F(\mathbb{Z})$, then $F(G) = k \operatorname{rank} G$ whenever G is a finitely generated abelian group.*

Proof. If G is a finitely generated abelian group, then the fundamental theorem of finitely generated abelian groups implies $G \cong \mathbb{Z}^n \oplus T$ for some $n \in \mathbb{Z}^+$ and some torsion group T. By properties (1), (2), and (4) of Definition 4.2.1, $F(G) = F(\mathbb{Z}^n \oplus T) = F(\mathbb{Z}^n) + F(T) = F(\mathbb{Z}^n) = nF(\mathbb{Z}) = (\operatorname{rank} G)F(\mathbb{Z}) = k \operatorname{rank} G$. □

Lemma 4.2.7. *If $F : \mathbf{Ab} \to \mathbb{Z}^+ \cup \{\infty\}$ is an exact size function, G is an abelian group with $F(G) = 0$, and H is a subgroup of G, then $F(H) = 0$ and $F(G/H) = 0$.*

Proof. Since the sequence $0 \to H \to G \to G/H \to 0$ is exact, we have $F(G) = F(H) + F(G/H)$. However, since $F(G) = 0$ and the values of $F(H)$ and $F(G/H)$ are non-negative, we must have $F(H) = 0$ and $F(G/H) = 0$. □

Example 4.2.8 (Examples of Exact Size Functions). Parts (i), (iii), and (iv) of Proposition 4.1.4 show that rank is an exact size function on the class of abelian groups. Furthermore, in analogy with (4.2), one can generalize this function as follows: If k is any field of characteristic 0, then k may be viewed as a $\mathbb{Z}$-module and for any abelian group G, the tensor product $\mathsf{k} \otimes_{\mathbb{Z}} G$ is a vector space over k. We may then define

$$\operatorname{rank}_{\mathsf{k}}(G) := \dim_{\mathsf{k}}(\mathsf{k} \otimes_{\mathbb{Z}} G)$$

where $\dim_{\mathsf{k}}$ denotes the dimension as a k-vector space.

It is straightforward to verify properties (1) and (2) of Definition 4.2.2, and property (3) of Definition 4.2.2 follows from the fact k is flat as a $\mathbb{Z}$-module. (Recall

that a $\mathbb{Z}$-module is flat if and only if it is torsion free.) Thus rank_{k} is an exact size function, and when $\mathsf{k} = \mathbb{Q}$ we recover the usual rank function.

Another example of an exact size function is

$$F(G) = \begin{cases} \infty & \text{if } G \text{ is not a torsion group} \\ 0 & \text{if } G \text{ is a torsion group.} \end{cases}$$

In particular, $F(\mathbb{Z}) = \infty$.

Example 4.2.9 (Examples of Size Functions). Parts (i) and (iii) of Proposition 4.1.9, Proposition 4.1.10, and Proposition 4.1.13 show that corank is a size function on the class of abelian groups, and Example 4.1.15 shows that corank is not an exact size function. If X is torsion-free, then $F(G) := \text{rank}(\text{Hom}(G, X))$ is a size function. We check property (2). Let $\phi : G \to X$ be a homomorphism and let G is a torsion group. If $n\phi = 0$, then $n\phi(g) = 0$ for all $g \in G$. Now, $\phi(g) \in X$, so $\phi(g) = 0$ for all $g \in G$ because X is torsion-free. Thus $\text{Hom}(G, X)) = 0$ and so $F(G) = 0$.

We have also that $F(G) := \text{rank}(\text{Hom}(G, X))$ is an exact size function if X is torsion-free and divisible. However, if X is an abelian group, then $F(G) := \text{rank}(\text{Hom}(G, X))$ need not be a size function in general. Moreover, if we take $F(G) := \text{rank}(\text{Hom}(G, \mathbb{Z}))$, then $F(G) = \text{corank}(G)$.

4.3 Size Functions and K-theory of unital Leavitt path algebras

4.3.1 Using exact size functions to determine the number of singular vertices

Theorem 4.3.1. *Suppose E is a graph with finitely many vertices, k is a field, and $F : \mathbf{Ab} \to \mathbb{Z}^+ \cup \{\infty\}$ is an exact size function (see Definition 4.2.2). If $n \in \mathbb{N}$ is a natural number for which $F(K_n(\mathsf{k})) < \infty$, and $0 < F(K_{n-1}(\mathsf{k})) < \infty$, then $F(K_n(L_\mathsf{k}(E))) < \infty$ and*

$$|E^0_{\text{sing}}| = \frac{(F(K_n(\mathsf{k})) + F(K_{n-1}(\mathsf{k}))) \operatorname{rank} K_0(L_\mathsf{k}(E)) - F(K_n(L_\mathsf{k}(E)))}{F(K_{n-1}(\mathsf{k}))}.$$

Proof. The long exact sequence of Theorem 3.1.1 induces the short exact sequence

$$\begin{aligned} 0 \longrightarrow \operatorname{coker}\left(\begin{pmatrix} B_E^t - I \\ C_E^t \end{pmatrix} : K_n(\mathsf{k})^{E^0_{\text{reg}}} \to K_n(\mathsf{k})^{E^0}\right) &\longrightarrow K_n(L_\mathsf{k}(E)) \\ \longrightarrow \ker\left(\begin{pmatrix} B_E^t - I \\ C_E^t \end{pmatrix} : K_{n-1}(\mathsf{k})^{E^0_{\text{reg}}} \to K_{n-1}(\mathsf{k})^{E^0}\right) &\longrightarrow 0. \end{aligned} \tag{4.4}$$

Since F is an exact size function, it follows from Proposition 4.2.4 that F is also a size function and satisfies the properties listed in Definition 4.2.1.

Theorem 3.2.4 implies that there exist $d_1, \ldots, d_k \in \{2, 3, \ldots\}$ and $m \in \mathbb{Z}^+$ such that

$$\begin{aligned} &\operatorname{coker}\left(\begin{pmatrix} B_E^t - I \\ C_E^t \end{pmatrix} : K_n(\mathsf{k})^{E^0_{\text{reg}}} \to K_n(\mathsf{k})^{E^0}\right) \\ &\quad \cong K_n(\mathsf{k}) \Big/ \langle d_1 x : x \in K_n(\mathsf{k}) \rangle \oplus \cdots \oplus K_n(\mathsf{k}) \Big/ \langle d_k x : x \in K_n(\mathsf{k}) \rangle \oplus K_n(\mathsf{k})^{m + |E^0_{\text{sing}}|} \end{aligned}$$

and

$$\ker\left(\begin{pmatrix} B_E^t - I \\ C_E^t \end{pmatrix} : K_{n-1}(\mathsf{k})^{E^0_{\mathrm{reg}}} \to K_{n-1}(\mathsf{k})^{E^0}\right) \cong K_{n-1}(\mathsf{k})^m \oplus \left(\bigoplus_{i=1}^{k} \ker((d_i) : K_{n-1}(\mathsf{k}) \to K_{n-1}(\mathsf{k}))\right)$$

and furthermore, m satisfies $\operatorname{rank} K_0(L_{\mathsf{k}}(E)) = m + |E^0_{\mathrm{sing}}|$. We may now use the fact that F breaks up over direct sums to evaluate F on the cokernel and kernel. Since $K_n(\mathsf{k})/\langle d_i x : x \in K_n(\mathsf{k})\rangle$ is a torsion group for all $1 \leq i \leq k$, the size function F assigns a value of zero to these groups, and since E^0 is finite, $K_n(\mathsf{k})^{m+|E^0_{\mathrm{sing}}|}$ is a finite direct sum and $F\left(K_n(\mathsf{k})^{m+|E^0_{\mathrm{sing}}|}\right) = (m+|E^0_{\mathrm{sing}}|)F(K_n(\mathsf{k})) = (\operatorname{rank} K_0(L_{\mathsf{k}}(E)))F(K_n(\mathsf{k}))$. Thus

$$F\left(\operatorname{coker}\left(\begin{pmatrix} B_E^t - I \\ C_E^t \end{pmatrix} : K_n(\mathsf{k})^{E^0_{\mathrm{reg}}} \to K_n(\mathsf{k})^{E^0}\right)\right) = (\operatorname{rank} K_0(L_{\mathsf{k}}(E)))\, F(K_n(\mathsf{k})). \quad (4.5)$$

In addition, since $\ker((d_i) : K_n(\mathsf{k}) \to K_n(\mathsf{k}))$ is a torsion group for all $1 \leq i \leq k$, the size function F assigns a value of zero to these groups, and since m is finite, $F(K_{n-1}(\mathsf{k})^m) = mF(K_{n-1}(\mathsf{k})) = (\operatorname{rank} K_0(L_{\mathsf{k}}(E)) - |E^0_{\mathrm{sing}}|)F(K_{n-1}(\mathsf{k}))$. Thus

$$F\left(\ker\left(\begin{pmatrix} B_E^t - I \\ C_E^t \end{pmatrix} : K_{n-1}(\mathsf{k})^{E^0_{\mathrm{reg}}} \to K_{n-1}(\mathsf{k})^{E^0}\right)\right) = (\operatorname{rank} K_0(L_{\mathsf{k}}(E)) - |E^0_{\mathrm{sing}}|)F(K_{n-1}(\mathsf{k})). \quad (4.6)$$

Since F is an exact size function, we may use (4.4), together with (4.5) and (4.6), to deduce

$$\begin{aligned} &F(K_n(L_{\mathsf{k}}(E))) \\ &\quad = (\operatorname{rank} K_0(L_{\mathsf{k}}(E)))\, F(K_n(\mathsf{k})) + (\operatorname{rank} K_0(L_{\mathsf{k}}(E)) - |E^0_{\mathrm{sing}}|)F(K_{n-1}(\mathsf{k})) \\ &\quad = (F(K_n(\mathsf{k})) + F(K_{n-1}(\mathsf{k})))\operatorname{rank} K_0(L_{\mathsf{k}}(E)) - |E^0_{\mathrm{sing}}|F(K_{n-1}(\mathsf{k})). \end{aligned}$$

Since $F(K_n(\mathsf{k})) < \infty$ and $F(K_{n-1}(\mathsf{k})) < \infty$ by hypothesis, and since rank $K_0(L_\mathsf{k}(E)) < \infty$, we have that $F(K_n(L_\mathsf{k}(E))) < \infty$. Also, we obtain

$$F(K_n(L_\mathsf{k}(E))) - (F(K_n(\mathsf{k})) + F(K_{n-1}(\mathsf{k}))) \operatorname{rank} K_0(L_\mathsf{k}(E)) = -|E^0_{\text{sing}}|F(K_{n-1}(\mathsf{k}))$$

and since $F(K_{n-1}(\mathsf{k})) > 0$ by hypothesis, we may divide to obtain

$$|E^0_{\text{sing}}| = \frac{(F(K_n(\mathsf{k})) + F(K_{n-1}(\mathsf{k}))) \operatorname{rank} K_0(L_\mathsf{k}(E)) - F(K_n(L_\mathsf{k}(E)))}{F(K_{n-1}(\mathsf{k}))}.$$

□

Corollary 4.3.2. *Suppose E and F are simple graphs with finitely many vertices and infinitely many edges, and suppose that k is a field. If there exist an exact size function $F : \mathbf{Ab} \to \mathbb{Z}^+ \cup \{\infty\}$ and a natural number $n \in \mathbb{N}$ for which $F(K_n(\mathsf{k})) < \infty$, and $0 < F(K_{n-1}(\mathsf{k})) < \infty$, then the following are equivalent:*

(i) $L_k(E)$ and $L_k(F)$ are Morita equivalent.

(ii) $K_0(L_k(E)) \cong K_0(L_k(F))$ and $K_n(L_k(E)) \cong K_n(L_k(F))$.

Proof. We obtain $(i) \implies (ii)$ from the fact that Morita equivalent algebras have algebraic K-theory groups that are isomorphic. For $(ii) \implies (i)$, we see that (ii) combined with Theorem 4.3.1 implies that $|E^0_{\text{sing}}| = |F^0_{\text{sing}}|$ and $K_0(L_\mathsf{k}(E)) \cong K_0(L_\mathsf{k}(F))$. It follows from [22, Theorem 7.4] that $L_\mathsf{k}(E)$ and $L_\mathsf{k}(F)$ are Morita equivalent. □

Remark 4.3.3. Since rank is an exact size function, both Theorem 4.3.1 and Corollary 4.3.2 apply when $F(-) = \operatorname{rank}(-)$.

Remark 4.3.4. Note that in both Theorem 4.3.1 and Corollary 4.3.2 the value of $n = 1$ is allowed. Also note that since $K_0(L_{\mathsf{k}}(F))$ is a finitely generated abelian group, we always have $F(K_0(L_{\mathsf{k}}(F))) = F(\mathbb{Z}) \operatorname{rank} K_0(L_{\mathsf{k}}(F))$ for any size function F, by Proposition 4.2.6.

4.3.2 Using size functions to determine the number of singular vertices

Theorem 4.3.5. *Suppose E is a graph with finitely many vertices, k is a field, and $F : \mathbf{Ab} \to \mathbb{Z}^+ \cup \{\infty\}$ is a size function (see Definition 4.2.1). If $n \in \mathbb{N}$ is a natural number for which $F(K_n(\mathsf{k})) = 0$, and $0 < F(K_{n-1}(\mathsf{k})) < \infty$, then*

$$|E^0_{\text{sing}}| = \operatorname{rank} K_0(L_{\mathsf{k}}(E)) - \frac{F(K_n(L_{\mathsf{k}}(E)))}{F(K_{n-1}(\mathsf{k}))}.$$

Proof. The long exact sequence of Theorem 3.1.1 induces the short exact sequence

$$\begin{aligned} 0 \longrightarrow \operatorname{coker}\left(\begin{pmatrix} B_E^t - I \\ C_E^t \end{pmatrix} : K_n(\mathsf{k})^{E^0_{\text{reg}}} \to K_n(\mathsf{k})^{E^0}\right) &\longrightarrow K_n(L_{\mathsf{k}}(E)) \\ \longrightarrow \ker\left(\begin{pmatrix} B_E^t - I \\ C_E^t \end{pmatrix} : K_{n-1}(\mathsf{k})^{E^0_{\text{reg}}} \to K_{n-1}(\mathsf{k})^{E^0}\right) &\longrightarrow 0 \end{aligned} \tag{4.7}$$

and Theorem 3.2.4 implies that there exist $d_1, \ldots, d_k \in \{2, 3, \ldots\}$ and $m \in \mathbb{Z}^+$ such that

$$\begin{aligned} &\operatorname{coker}\left(\begin{pmatrix} B_E^t - I \\ C_E^t \end{pmatrix} : K_n(\mathsf{k})^{E^0_{\text{reg}}} \to K_n(\mathsf{k})^{E^0}\right) \\ &\quad \cong K_n(\mathsf{k}) \Big/ \langle d_1 x : x \in K_n(\mathsf{k}) \rangle \oplus \cdots \oplus K_n(\mathsf{k}) \Big/ \langle d_k x : x \in K_n(\mathsf{k}) \rangle \oplus K_n(\mathsf{k})^{m + |E^0_{\text{sing}}|} \end{aligned}$$

and

$$\ker\left(\begin{pmatrix} B_E^t - I \\ C_E^t \end{pmatrix} : K_{n-1}(\mathsf{k})^{E^0_{\text{reg}}} \to K_{n-1}(\mathsf{k})^{E^0}\right)$$

$$\cong K_{n-1}(\mathsf{k})^m \oplus \left(\bigoplus_{i=1}^{k} \ker((d_i) : K_{n-1}(\mathsf{k}) \to K_{n-1}(\mathsf{k})) \right)$$

and furthermore, m satisfies

$$\operatorname{rank} K_0(L_{\mathsf{k}}(E)) = m + |E^0_{\text{sing}}|.$$

We may now use the fact that F breaks up over direct sums to evaluate F on the cokernel and kernel. Since $K_n(\mathsf{k})/\langle d_i x : x \in K_n(\mathsf{k})\rangle$ is a torsion group for all $1 \leq i \leq k$, the size function F assigns a value of zero to these groups, and since E^0 is finite and $F(K_n(\mathsf{k})) = 0$ by hypothesis, we may conclude that $F\left(K_n(\mathsf{k})^{m+|E^0_{\text{sing}}|}\right) = 0$. Thus

$$F\left(\operatorname{coker}\left(\begin{pmatrix} B_E^t - I \\ C_E^t \end{pmatrix} : K_n(\mathsf{k})^{E^0_{\text{reg}}} \to K_n(\mathsf{k})^{E^0}\right)\right) = 0. \tag{4.8}$$

In addition, since $\ker((d_i) : K_n(\mathsf{k}) \to K_n(\mathsf{k}))$ is a torsion group for all $1 \leq i \leq k$, the size function F assigns a value of zero to these groups, and since m is finite, $F(K_{n-1}(\mathsf{k})^m) = mF(K_{n-1}(\mathsf{k})) = (\operatorname{rank} K_0(L_{\mathsf{k}}(E)) - |E^0_{\text{sing}}|)F(K_{n-1}(\mathsf{k}))$. Thus

$$F\left(\ker\left(\begin{pmatrix} B_E^t - I \\ C_E^t \end{pmatrix} : K_{n-1}(\mathsf{k})^{E^0_{\text{reg}}} \to K_{n-1}(\mathsf{k})^{E^0}\right)\right) = (\operatorname{rank} K_0(L_{\mathsf{k}}(E)) - |E^0_{\text{sing}}|)F(K_{n-1}(\mathsf{k})). \tag{4.9}$$

Using the short exact exact sequence in (4.7), Lemma 4.2.5, and the computations in (4.8) and (4.9), Moreover, this equation together with the hypothesis that $F(K_{n-1}(\mathsf{k})) < \infty$ and the fact that $\operatorname{rank} K_0(L_{\mathsf{k}}(E)) < \infty$ implies $F(K_n(L_{\mathsf{k}}(E))) < \infty$. In addition, since $0 < F(K_{n-1}(\mathsf{k})) < \infty$ by hypothesis and since $\operatorname{rank} K_0(L_{\mathsf{k}}(E)) < \infty$, we may divide to obtain

$$\frac{F(K_n(L_{\mathsf{k}}(E)))}{F(K_{n-1}(\mathsf{k}))} = \operatorname{rank} K_0(L_{\mathsf{k}}(E)) - |E^0_{\text{sing}}|,$$

and thus $|E^0_{\text{sing}}| = \operatorname{rank} K_0(L_{\mathsf{k}}(E)) - \frac{F(K_n(L_{\mathsf{k}}(E)))}{F(K_{n-1}(\mathsf{k}))}$. □

Remark 4.3.6. Although Theorem 4.3.1 and Theorem 4.3.5 are similar, neither implies the other. The hypotheses of Theorem 4.3.1 require F to be an exact size function, while Theorem 4.3.5 allows F to be any size function. Furthermore, the hypotheses of Theorem 4.3.5 require $F(K_n(\mathsf{k})) = 0$, while Theorem 4.3.1 only requires $F(K_n(\mathsf{k}))$ to be finite. Thus the hypotheses of Theorem 4.3.1 impose stronger conditions on the properties of F, while the hypotheses of Theorem 4.3.5 impose stronger conditions on the value $F(K_n(\mathsf{k}))$.

Corollary 4.3.7. *Suppose E and F are simple graphs with finitely many vertices and an infinite number of edges, and suppose that k is a field. If there exist a size function $F : \mathbf{Ab} \to \mathbb{Z}^+ \cup \{\infty\}$ and a natural number $n \in \mathbb{N}$ for which $F(K_n(\mathsf{k})) = 0$, and $0 < F(K_{n-1}(\mathsf{k})) < \infty$, then the following are equivalent:*

(i) $L_k(E)$ and $L_k(F)$ are Morita equivalent.

(ii) $K_0(L_k(E)) \cong K_0(L_k(F))$ and $K_n(L_k(E)) \cong K_n(L_k(F))$.

Proof. We obtain $(i) \implies (ii)$ from the fact that Morita equivalent algebras have algebraic K-theory groups that are isomorphic. For $(ii) \implies (i)$, we see that (ii) combined with Theorem 4.3.5 implies that $|E^0_{\text{sing}}| = |F^0_{\text{sing}}|$ and $K_0(L_\mathsf{k}(E)) \cong K_0(L_\mathsf{k}(F))$. It follows from [22, Theorem 7.4] that $L_\mathsf{k}(E)$ and $L_\mathsf{k}(F)$ are Morita equivalent. □

Remark 4.3.8. Since corank is a size function, Theorem 4.3.5 and Corollary 4.3.7 apply when $F(-) = \text{corank}(-)$.

Remark 4.3.9. Note that in both Theorem 4.3.5 and Corollary 4.3.7 the value of $n = 1$ is allowed. Also note that since $K_0(L_\mathsf{k}(F))$ is a finitely generated abelian

group, we always have $F(K_0(L_{\mathsf{k}}(F))) = F(\mathbb{Z}) \operatorname{rank} K_0(L_{\mathsf{k}}(F))$ for any size function F, by Proposition 4.2.6.

Remark 4.3.10. It was proven in [22, Theorem 7.4] that if E is a simple graph with a finite number of vertices and an infinite number of edges, then $(K_0(L_{\mathsf{k}}(E)), |E^0_{\text{sing}}|)$ is a complete Morita equivalence invariant for $L_{\mathsf{k}}(E)$. Moreover, it was proven in [22, Corollary 6.14] that if k is a field with no free quotients, then $|E^0_{\text{sing}}|$ is determined by the pair $(K_0(L_{\mathsf{k}}(E)), K_1(L_{\mathsf{k}}(E)))$, and hence $(K_0(L_{\mathsf{k}}(E)), K_1(L_{\mathsf{k}}(E)))$ is a complete Morita equivalence invariant for $L_{\mathsf{k}}(E)$ in this case. Since a field k has no free quotients if and only if the abelian group $K_1(\mathsf{k}) \cong \mathsf{k}^\times$ has no free quotients, we see that k has no free quotients if and only if $\operatorname{corank} K_1(\mathsf{k}) = 0$. Thus the result from [22] is a special case of Corollary 4.3.7 when $n = 1$ and $F(-) = \operatorname{corank}(-)$.

4.3.3 Number Fields

A *number field* is a finite field extension of $\mathbb{Q}$. (We note that, in particular, $\mathbb{Q}$ itself is considered a number field.) If k is a number field of degree n over $\mathbb{Q}$, then by the primitive element theorem we may write $\mathsf{k} = \mathbb{Q}(\alpha)$ for an element α of degree n. If we let $p(x)$ be the minimal polynomial of α, then since $\mathbb{Q}$ has characteristic zero, $p(x)$ is separable and we may factor the polynomial $p(x)$ into n monomials with distinct roots. These roots will appear as distinct real numbers together with distinct conjugate pairs, and we write

$$p(x) = (x - \lambda_1) \dots (x - \lambda_{r_1})(x - \mu_1)(x - \overline{\mu}_1) \dots (x - \mu_{r_2})(x - \overline{\mu}_{r_2})$$

for distinct elements $\lambda_1, \ldots, \lambda_{r_1} \in \mathbb{R}$ and $\mu_1, \ldots, \mu_{r_2} \in \mathbb{C} \setminus \mathbb{R}$. Moreover, if we let $q_i(x) := (x - \mu_i)(x - \overline{\mu}_i)$ for $1 \leq i \leq r_2$ be the degree 2 polynomial in $\mathbb{R}[x]$ with μ_i and $\overline{\mu}_i$ as roots, then $p(x) = (x - \lambda_1) \ldots (x - \lambda_{r_1}) q_1(x) \ldots q_{r_2}(x)$ is a factorization of $p(x)$ into irreducible factors over $\mathbb{R}$. If we tensor k with $\mathbb{R}$, we may use the Chinese remainder theorem to obtain

$$
\begin{aligned}
&\mathsf{k} \otimes_{\mathbb{Q}} \mathbb{R} \\
&\quad \cong \mathbb{Q}(\alpha) \otimes_{\mathbb{Q}} \mathbb{R} \\
&\quad \cong (\mathbb{Q}[x]/\langle p(x) \rangle) \otimes_{\mathbb{Q}} \mathbb{R} \\
&\quad \cong \mathbb{R}[x]/\langle p(x) \rangle \\
&\quad \cong \mathbb{R}[x]/\langle (x - \lambda_1) \ldots (x - \lambda_{r_1}) q_1(x) \ldots q_{r_2}(x) \rangle \\
&\quad \cong \mathbb{R}[x]/\langle (x - \lambda_1) \rangle \times \ldots \times \mathbb{R}[x]/\langle (x - \lambda_{r_1}) \rangle \times \mathbb{R}[x]/\langle q_1(x) \rangle \times \ldots \times \mathbb{R}[x]/\langle q_{r_2}(x) \rangle \\
&\quad \cong \mathbb{R}(\lambda_1) \times \ldots \times \mathbb{R}(\lambda_{r_1}) \times \mathbb{R}(\mu_1) \times \ldots \times R(\mu_{r_2}) \\
&\quad \cong \mathbb{R}^{r_1} \times \mathbb{C}^{r_2}.
\end{aligned}
$$

Since r_1 is the number of real roots of $p(x)$ and r_2 is the number of conjugate pairs of non-real roots of $p(x)$, we have that $r_1, r_2 \in \mathbb{Z}^+$ and $r_1 + 2r_2 = n$. We observe (and this will be useful for us later) that at least one of r_1 and r_2 is strictly positive. The non-negative integer r_1 is called *the number of real places of* k, and the non-negative integer r_2 is called *the number of complex places of* k. If $r_2 = 0$, then k is said to be *totally real*, and if $r_1 = 0$, then k is said to be *totally complex*.

Theorem 4.3.11 (Theorem 1.5 of [13] or Theorem IV.1.18 of [27])**.** *Let* k *be a number*

field with r_1 real places and r_2 complex places. Then for $n \in \mathbb{Z}^+$,

$$\operatorname{rank} K_n(\mathsf{k}) = \begin{cases} 1 & n = 0 \\ \infty & n = 1 \\ 0 & n = 2k \text{ and } k > 0 \\ r_1 + r_2 & n = 4k+1 \text{ and } k > 0 \\ r_2 & n = 4k+3 \text{ and } k \geq 0. \end{cases}$$

In particular, $K_{6+4k}(\mathsf{k})$ is a torsion group and $K_{5+4k}(\mathsf{k})$ has strictly positive finite rank for any $k \in \mathbb{Z}^+$.

Theorem 4.3.12. *Let k be a number field with r_1 real places and r_2 complex places. If E is a graph with finitely many vertices, then for any $k \in \mathbb{Z}^+$*

$$|E^0_{\text{sing}}| = \operatorname{rank}(K_0(L_{\mathsf{k}}(E))) - \frac{\operatorname{rank}(K_{6+4k}(L_{\mathsf{k}}(E)))}{r_1 + r_2}.$$

In addition, if E and F are simple graphs with finitely many vertices and an infinite number of edges, then the following are equivalent:

(i) $L_k(E)$ and $L_k(F)$ are Morita equivalent.

(ii) $K_0(L_k(E)) \cong K_0(L_k(F))$ and $K_{6+4k}(L_k(E)) \cong K_{6+4k}(L_k(F))$ for all $k \in \mathbb{Z}^+$.

(iii) $K_0(L_k(E)) \cong K_0(L_k(F))$ and $K_{6+4k}(L_k(E)) \cong K_{6+4k}(L_k(F))$ for some $k \in \mathbb{Z}^+$.

In addition, if k is not totally real (i.e., $r_2 \neq 0$), then whenever E is a graph with finitely many vertices and $k \in \mathbb{Z}^+$ we have

$$|E^0_{\text{sing}}| = \operatorname{rank}(K_0(L_{\mathsf{k}}(E))) - \frac{\operatorname{rank}(K_{4+4k}(L_{\mathsf{k}}(E)))}{r_2}.$$

Moreover, if k *is not totally real (i.e.,* $r_2 \neq 0$*), then whenever* E *and* F *are simple graphs with finitely many vertices and an infinite number of edges, the following are equivalent:*

(i) $L_k(E)$ *and* $L_k(F)$ *are Morita equivalent.*

(ii) $K_0(L_k(E)) \cong K_0(L_k(F))$ *and* $K_{4+2k}(L_k(E)) \cong K_{4+2k}(L_k(F))$ *for any* $k \in \mathbb{Z}^+$.

(iii) $K_0(L_k(E)) \cong K_0(L_k(F))$ *and* $K_{4+2k}(L_k(E)) \cong K_{4+2k}(L_k(F))$ *for some* $k \in \mathbb{Z}^+$.

Proof. Proposition 4.1.4 and Proposition 4.2.4 imply that rank$(-)$ is a size function. We now apply Theorem 4.3.5 and Corollary 4.3.7 noting that Theorem 4.3.11 implies that $\operatorname{rank} K_{2k}(\mathsf{k}) = 0$, $\operatorname{rank} K_{3+4k}(\mathsf{k}) = r_2$, and $\operatorname{rank} K_{5+4k}(\mathsf{k}) = r_1 + r_2$. □

Remark 4.3.13. Let E be a simple graph with finitely many vertices and an infinite number of edges. Theorem 4.3.12 shows that if k is a number field, then the pair $(K_0(L_{\mathsf{k}}(E)), K_6(L_{\mathsf{k}}(E)))$ is a complete Morita equivalence invariant for $L_{\mathsf{k}}(E)$, and if k is not totally real, then the pair $(K_0(L_{\mathsf{k}}(E)), K_4(L_{\mathsf{k}}(E)))$ is a complete Morita equivalence invariant for $L_{\mathsf{k}}(E)$.

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